ECE 301 (Section 001) Homework 9
Spring 2025, Dr. Chau-Wai Wong
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Problem 1 (Linear Regression in Matrix–Vector Form) An ECE student John is doing an electronic circuits lab during which he needs to determine the conductance of a resistor using a voltage meter, a current meter, and a DC power source. The voltage meter is connected in parallel with the resistor and the current meter is in series with the resistor. Both meters are analog devices so the readings recorded by John have errors. The power source is tunable and has a range of 1 to 5 V. Each time John will try a uniformly random input voltage level and record the readings of both voltage and current meters. Denote the voltage reading as x_i and the current reading as y_i for the ith measurement. Assume the true conductance $G = 2\ \text{m}\Omega^{-1}$.

a) Using a linear model $y_i = Gx_i + e_i$, where e_i is a zero-mean noise with standard deviation $\sigma_e = 0.1$ mA, simulate a dataset of $n = 10$ measurements. [Hint: You may use Matlab command `rand()` to generate a uniformly random value in $[0, 1]$ and `0.1*randn()` to generate a zero-mean Gaussian noise with standard deviation 0.1. Throughout this problem, you may ignore the units such as $\text{m}\Omega^{-1}$ and mA and focus only on the numbers.]

b) Express the linear model in a matrix-vector form. Clearly indicate the matrices/vectors $\mathbf{y}, \mathbf{X}, \boldsymbol{\beta}$, and $\mathbf{e}$. Directly implement the formula of the least-squares (LS) estimator, $\hat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$, into a computer function that takes as two input vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, and output a number $\hat{G}$. Apply your function to the simulated data. What is the value of $\hat{G}$? [Hint: $\mathbf{X}$ is a n-by-1 "matrix," and $\boldsymbol{\beta}$ is a 1-by-1 "vector."]

c) John's friend, Tom, proposed a more intuitive estimator for the conductance: $\tilde{G} = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{x_i}$. Let's call it "Tom's estimator" for convenience. Write a computer function that takes as two input vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, and output a number $\tilde{G}$. Apply your function to the simulated data. What is the value of $\tilde{G}$?

d) Generate 1,000 datasets. Repeatedly apply function written in (b) and collect 1,000 LS estimates and calculate the sample variance of these 1,000 values.

e) Use the 1,000 datasets generated in (d). Repeatedly apply function written in (c) and collect 1,000 Tom's estimates and calculate the sample variance of these 1,000 values. You should find that the LS estimator has a smaller variance than Tom's estimator.

Problem 2 (Fourier Series Review: DC Power Supply) One technique for building a DC power supply is to take an AC signal and full-wave rectify it. That is, we put the AC signal $x(t)$ through a system that produces $y(t) = |x(t)|$ as its output.

a) Sketch the input and output waveforms if $x(t) = \cos(t)$. What are the fundamental periods of the input and the output?

b) If $x(t) = \cos(t)$, determine the coefficients of the Fourier series for the output $y(t)$.

c) What is the amplitude of the DC component of the input signal?

d) What is the amplitude of the DC component of the output signal?

Problem 3 (Fourier Series Review: Analysis/Forward Transform) Find the Fourier series coefficients for each of the following, given that $x(t)$ is a periodic function with period 2π.

a)

$$x(t) = t^3, \quad t \in [-\pi, \pi].$$

Hint:

$$\int t^3 e^{-j\omega kt} dt = \frac{e^{-jkt\omega}\left(jk^3t^3\omega^3 + 3k^2t^2\omega^2 - 6jkt\omega - 6\right)}{k^4\omega^4} + C. \tag{1}$$

b)

$$x(t) = |t|, \quad t \in [-\pi, \pi].$$

Hint: i) The absolute sign goes away when the domain is split into the positive and the negative halves. ii) You will need to use integration by parts.

c) (2', optional) Prove equation (1).

Problem 4 (Eigen-signals/functions of an LTI System) In the lecture, it was stated without proof that $e^{j\omega_0 t}$ is an eigen-signal of an LTI system $h(t)$. In other words, when the LTI system operates on an input signal $x(t) = e^{j\omega_0 t}$ [or $x(t)$ is sent into the LTI system], the output $y(t)$ is merely a scaled version of $x(t)$ for all $t \in \mathbb{R}$. Show that the scaling factor is

$$H(j\omega_0) = \int_{-\infty}^{\infty} h(t)e^{-j\omega_0 t}\, dt. \tag{2}$$

Recall that the input–output relation of an LTI system is related by $y(t) = h(t) * x(t)$.

Group Study (1', bonus) In-Person: Take a selfie with all group members' faces in the photo. Capture in the photo the homework assignment sheet that you are working on. Zoom: Take a screenshot of the whole team with everyone's webcam capturing his/her face. One of you will share the screen showing the specific homework assignment sheet.

Include the screenshot/selfie in your own homework submission as the last "problem." Your screenshot/selfie gets you 1 bonus point; your group members need to do it separately to earn their bonus points.