ECE 301 (Section 001) Homework 3, Spring 2021

Problem 1 (Impulse Function and Step Function)

a) Find and sketch the first derivatives of the following signals:

(i) $x(t) = u(t) - u(t-a)$, $a > 0$,

(ii) $x(t) = t[u(t) - u(t-a)]$, $a > 0$.

(iii) $x(t) = \text{sign}(t) = \begin{cases} 1, & t \geq 0, \\ -1, & t < 0. \end{cases}$

b) Given that the $\delta(\cdot)$ is the unit impulse function, prove the following property:

$$\int_{-\infty}^{\infty} \delta(ax)dx = \int_{-\infty}^{\infty} \frac{\delta(t)}{|a|}dt = \frac{1}{|a|},$$

where a is a scalar. Can you claim that $\delta(ax) = \frac{1}{|a|}\delta(t)$? If yes, in what sense? If no, why.

Problem 2 (Properties of Discrete Systems) Determine (with justification) whether the following systems are (i) memoryless, (ii) causal, (iii) invertible, (iv) stable, and (v) time invariant. For invertibility, either find an inverse system or an example of two inputs that lead to the same output. Note that $y[n]$ denotes the system output and $x[n]$ denotes the system input.

a) $y[n] = x[n]\,x[n-1]\,x[n+1]$

b) $y[n] = \begin{cases} x[n-1], & n \leq 0, \\ x[n+1], & n > 0. \end{cases}$

c) $y[n] = \cos(x[n])$

Problem 3 (Properties of Continuous Systems) Determine (with justification) whether the following systems are (i) memoryless, (ii) causal, (iii) invertible, (iv) stable, and (v) time invariant. For invertibility, either find an inverse system or an example of two inputs that lead to the same output. Note that $y(t)$ denotes the system output and $x(t)$ is the system input.

a) $y(t) = (t-2)\,x(t+2)$

b) $y(t) = x(t-1) + x(4-t)$

c) $y(t) = x\big(\sin(t)\big)$

d) (Optional, not graded) $y(t) = \begin{cases} 0, & x(t) < 1, \\ \int_0^1 x(t-\tau)d\tau, & x(t) \geq 1. \end{cases}$

Problem 4 (Interconnected Systems)

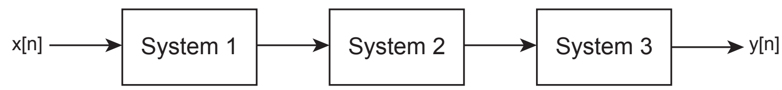
a) Consider three systems with these input–output relationships:

$$\text{System 1 (upsampling):} \quad w[n] = \begin{cases} x[n/3], & n \text{ multiple of 3,} \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{System 2 (smoothing):} \quad z[n] = w[n] + 2w[n-1] + w[n-2].$$

$$\text{System 3 (downsampling):} \quad y[n] = z[3n].$$

Suppose that these systems are connected in series, as shown below.



Find the input–output relationship, namely, write $y[n]$ as a function of $x[n]$, for the overall interconnected system. Is it time invariant? (Hint: You can draw plots to see "physically," how a three-sample input signal, e.g., $x[0] = 1$, $x[1] = -1$, $x[2] = 2$, $x[n] = 0$ for all other n, evolves as it passes each block of the system.)

b) (4') (i) Please write down your Quiz 1 grade ranges from 0 to 4.

(ii) If your grade is less than 4', you may recover at most 1' by drawing $s^{-1}\text{rect}(t/s)$ for $s \in \{0.1, 1, 10\}$ on the same graph by hand. Please properly label the width and height of each rect$(\cdot)$ function.

Problem 5 (Delta Function as a Limiting Function)

a) A resistor of an RC circuit has an (output) voltage of $y_{t_0}(t) = \left[1 - e^{-5(t-t_0)}\right] u(t-t_0)$ supplied by an (input) voltage source with a unit-step voltage of $x(t) = u(t - t_0)$.

(i) Without going into mathematical derivation/manipulation, argue why the RC circuit can be viewed as a time-invariant system.

(ii) Suppose the input voltage is 1 during the interval of $t = 0$ to 0.5 seconds and 0 otherwise, i.e., $x(t) = u(t) - u(t-0.5)$. Using the the superposition principle from the circuit theory, show that the analytic form of the output voltage is

$$y(t) = \begin{cases} 1 - e^{-5t}, & 0 \leq t \leq 0.5, \\ 11.18e^{-5t}, & t > 0.5, \\ 0, & \text{else.} \end{cases}$$

(iii) How is the superposition principle related to the linearity property of the system of the RC circuit? (Try this problem after Monday's lecture.)

(iv) What are the analytic forms of the output voltage given the following input voltage?

(1) $x(t) = 2[u(t) - u(t - 0.5)]$

(2) $x(t) = 4[u(t) - u(t - 0.25)]$

(3) $x(t) = 20[u(t) - u(t - 0.05)]$

(v) Using the same amplitude scale and time scale on the same Matlab plot, visualize the output voltages in (iv) over $0 \leq t \leq 1$. Note that, using (iv)(1) as an example, you can either draw a piecewise function separately for `t = 0 : 0.01 : 0.5` and `t = 0.5 : 0.01 : 1` or in one piece for `t = 0 : 0.01 : 1`.

b) (5', bonus) You should notice that the sequence of curves you plotted for a)(v) are converging. Mathematically prove that the limiting curve is the delta function.