ECE 301 (Section 001) Homework 7, Spring 2021

Problem 1 (Projection and Orthogonal Matrices)

a) Quiz 2 Follow-up (4')

(i) Please attach the screenshot of your Quiz 2 grade.

(ii) If your score is less than 4' or 100% (including those who missed the quiz), you may recover at most 2' by completing the following question.

What's the resulting projected *vector* $\hat{\mathbf{x}}$ when $\mathbf{x} = \sqrt{2}[1.4, 2.4]$ is projected onto vector $\mathbf{w} = [1, -1]$? What's the *norm* of the error/residual vector $\mathbf{x} - \hat{\mathbf{x}}$?

b) You have learned in class that a real-valued *orthogonal matrix* $\mathbf{P}$ has the property that $\mathbf{P}^T\mathbf{P} = \mathbf{P}\mathbf{P}^T = \mathbf{I}$. In signal processing, often times signals are represented in complex values. We can define for a complex-valued square matrix a similar concept called the *unitary matrix*. A unitary matrix $\mathbf{Q}$ satisfies the property that $\mathbf{Q}^H\mathbf{Q} = \mathbf{Q}\mathbf{Q}^H = \mathbf{I}$, where "^{H}" is the Hermitian operator that is a combination of both the transpose, "^{T}," and the complex conjugate, "$*$."

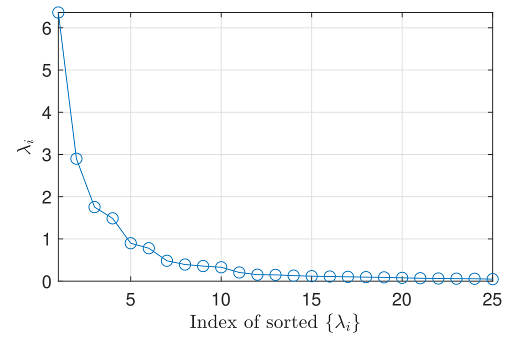
(i) The discrete Fourier transform (DFT) matrix for a signal of length 3 is defined as $\mathbf{Q}_3 = [q_{kn}]_{k,n\in\{0,1,2\}}$, where $q_{kn} = \frac{1}{\sqrt{3}}\exp(-j\frac{2\pi}{3}kn)$. Explicitly write out this 3-by-3 matrix. Simplify each entry but do not evaluate numerically the complex exponentials.

(ii) Verify that $\mathbf{Q}_3$ is a unitary matrix.

Problem 2 (PCA on Downsampled Yale Face Database) In this problem, we will explore PCA as a visualization tool for Yale Face Database. Download the `.m` files and the database. Extract the face image files into a folder named `yalefaces` and put the `.m` files at the same level of the folder. Open `main_pca_visualization.m` in Matlab.

a) Run the code of (a), explain the data structure of variable `img_buffer`. Set `preview_img_flag` to `1`, re-run the code to visually inspect the whole database.

b) Complete Matlab function `[V, Lambda_mat] = PcaViaKlt(data)` by implementing PCA using eigendecomposition on a sample covariance matrix of the face data. The detailed information about the input and outputs are given in the comments of the incomplete function. You may use built-in function `eig` for eigendecomposition. If your implementation is correct, after running the code of b), you will obtain a plot similar to the following.



c) Run the code of c) to visualize a few dominating eigenvectors. Comment on whether they reflect some characteristics of the faces you saw in a).

d) The code of d) projects each face image (coming from one of the four selected classes) onto a 2D space. Comment on PCA's data visualization performance in this specific example.

Problem 3 (Linear Regression in Matrix-Vector Form) An ECE student John is doing an electronic circuits lab during which he needs to determine the conductance of a resistor using a voltage meter, a current meter, and a DC power source. The voltage meter is connected in parallel with the resistor and the current meter is in series with the resistor. Both meters are analog devices so the readings recorded by John have errors. The power source is tunable and has a range of 1 to 5 V. Each time John will try a uniformly random input voltage level and record the readings of both voltage and current meters. Denote the voltage reading as x_i and the current reading as y_i for the ith measurement. Assume the true conductance $G = 2\ \text{m}\Omega^{-1}$.

a) Using a linear model $y_i = Gx_i + e_i$, where e_i is a zero-mean noise with standard deviation $\sigma_e = 0.1$ mA, simulate a dataset of $n = 10$ measurements. [Hint: You may use `rand()` to generate a uniformly random value in $[0, 1]$ and `0.1*randn()` to generate a zero-mean Gaussian noise with standard deviation 0.1. Throughout this problem, you may ignore the units such as $\text{m}\Omega^{-1}$ and mA and focus only on the numbers.]

b) Express the linear model in a matrix-vector form. Clearly indicate the matrices/vectors $\mathbf{y}, \mathbf{X}, \boldsymbol{\beta}$, and $\mathbf{e}$. Directly implement the formula of the least-squares (LS) estimator, $\hat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$, into a computer function that takes as two input vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, and output a number $\hat{G}$. Apply your function to the simulated data. What is the value of $\hat{G}$? [Hint: $\mathbf{X}$ is a n-by-1 "matrix," and $\boldsymbol{\beta}$ is a 1-by-1 "vector."]

c) John's friend, Tom, proposed a more intuitive estimator for the conductance: $\tilde{G} = \frac{1}{n}\sum_{i=1}^{n} \frac{y_i}{x_i}$. Let's call it "Tom's estimator" for convenience. Write a computer function that takes as two input vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, and output a number $\tilde{G}$. Apply your function to the simulated data. What is the value of $\tilde{G}$?

d) Generate 1,000 datasets. Repeatedly apply function written in (b) and collect 1,000 LS estimates and calculate the sample variance of these 1,000 values.

e) Use the 1,000 datasets generated in (d). Repeatedly apply function written in (c) and collect 1,000 Tom's estimates and calculate the sample variance of these 1,000 values. You should find that the LS estimator has a smaller variance than Tom's estimator.

Problem 4 (Machine Learning Intro Video) Watch this 10-minute video:

`https://youtu.be/YH1gdd8kJXo`

Write 6–8 sentences to concisely summarize machine learning and/or artificial intelligence.

Group Study (1', bonus) Take a screenshot of the whole team with everyone's camera capturing his/her face. One of you will share a window showing the specific homework assignment sheet that you are working on. Include the screenshot in your own homework submission as Problem 5. Your screen shoot gets you 1 bonus point; your group members need to do it separately to earn theirs.