ECE 301 (001) Linear Systems
Spring 2021 Midterm Exam 1
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This is a closed-book, closed-lecture notes exam. One-sided, letter-sized, handwritten cheat sheet is allowed. Calculators are not allowed. You need to answer only ***four complete problems***. Submission instructions:

1. You have a 15-minute time window to submit your answers to Gradescope. There will be a 10-point penalty for every late minute.
2. Use a scanner app to take a picture for every answer sheet. Do not forget to scan those pages on the back if you wrote answers on them.
3. The answer sheet for the unanswered problem should be kept empty and properly scanned.
4. Double check the scanned pages one by one. Make sure there is no missing page.
5. Upload your scanned document as a PDF file to Gradescope.

Problem 1 (25 pts) [Geometric Series] Let

$$S = q^m + q^{m+1} + \cdots + q^{n+3},$$

where $m, n \in \mathbb{Z}$, and $m < n$.

(a) Derive the analytic form of S without the help of the geometric series formula. Show intermediate steps clearly. Remember to deal with different cases separately.

(b) What happens when $n \to \infty$? Again, you may want to address different cases separately.

Problem 2 (25 pts) [Operations on Signals]

(a) You are given a discrete-time signal $x[n] = \delta[n+2] - 2\delta[n] + 3\delta[n-3] - \delta[n-5]$, $n \in \mathbb{Z}$. Sketch carefully the following signals.

$$y_1[n] = x[n/3].$$

Partial points will be considered for intermediate calculations/manipulations you write down.

(b) You are given signal $x(t) = 3\cos(t)\,[u(t) - u(t-2\pi)]$, $t \in \mathbb{R}$.

(i) Sketch $x(t)$. Clearly provide important tick values on the t-axis and x-axis to facilitate a precise graphical description of $x(t)$.

(ii) Calculate the first derivative of $x(t)$. Note that $(uv)' = u'v + uv'$ and $(\cos x)' = -\sin x$.

(iii) Sketch the first derivative of $x(t)$ using (1) the result of (ii), or (2) your observation from the graph you sketched for (i). If you proceed via route (2), justify in words how you obtained the various components in your plot.

Problem 3 (25 pts) [Periodicity and Sifting Property]

(a) Determine whether or not the following signals are periodic. If periodic, determine its fundamental period, otherwise explain why it is aperiodic. (If you use off-the-shelf formulas, make sure they are applied correctly. You may also try to find the period by starting from the definition, which allows us to give you partial credits even if the final result is wrong.)

(i) $x(t) = \cos(t/4 + \pi/6)$.

(ii) $x[n] = \cos\left(\frac{2022}{2021}\pi n\right) + \exp\left(j\frac{7}{2}\pi n\right)$.

(b) Simplify/evaluate the following expressions. To save your time, there is no need to copy the following equations to your answer sheet.

(i) $\sin(3k\omega)\delta(\omega)/\omega$.

(ii) $\sum_{n=-\infty}^{\infty} \delta[n+3]e^{n-3}\tan\left(\frac{\pi}{12}(6+n)\right)$.

Problem 4 (25 pts) [Time-Invariance and Linearity] Determine whether the following systems with input x and output y are (1) time-invariant, and whether they are (2) linear. You must justify with mathematical expressions and/or in words at each intermediate step.

(a) $y(t) = x\big((t-1)^2\big)$.

(b) $y(t) = x(t)\cos(\omega t)$.

Problem 5 (25 pts) [Convolution] You are given $x(t) = e^{-3t}u(t)$ and $h(t) = u(4-t)$ for $t \in \mathbb{R}$.

(a) Using a graphical approach to determine how the interval $t \in \mathbb{R}$ should be divided into subintervals on which $y(t) = x(t) * h(t)$ will have different analytic expressions. Explicitly specify the range of each subinterval. Note that the union of all subintervals must be $\mathbb{R}$.

(b) Evaluate $y(t)$ for $t \in \mathbb{R}$. You need to simplify the expressions for $y(t)$ on each subinterval of t.