ECE 301 (001) Linear Systems
Spring 2021 Midterm Exam 2
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This is a closed-book, closed-lecture notes exam. One-sided, letter-sized, handwritten cheat sheet is allowed. Calculators are not allowed. You need to answer only ***four complete problems***. Submission instructions:

1. You have a 15-minute time window to submit your answers to Gradescope. There will be a 10-point penalty for every late minute.
2. Use a scanner app to take a picture for every answer sheet. Do not forget to scan those pages on the back if you wrote answers on them.
3. The answer sheet for the unanswered problem should be kept empty and properly scanned. Only four problems will be graded and counted toward the grade.
4. Double check the scanned pages one by one. Make sure there is no missing page.
5. Upload your scanned document as a PDF file to Gradescope.

Problem 1 (25 pts) [Basis and Vector Space] You are given a vector space $V = \text{span}\{[1, -2, 0], [-1, 0, 3]\}$.

(a) Express V in a set representation.

(b) Can you find a basis for V?

(c) Are $[1, 2, 1]$, $[1, -4, 3]$, and $[1, 2, 3]$ in vector space V? Is yes, what are the coefficient for each vector of the basis you found in (b)?

(d) Illustrate vector space V using a plane formed by the vectors of the basis.

Problem 2 (25 pts) [Linear Regression in Matrix-Vector Form] An ECE student named Tom plans to test the fuel economy of his car in terms of how many gallons is needed for driving one mile. He will do four 4 test drives of x_i miles each, $i = 1, \cdots, 4$, and will measure the corresponding gas consumption Y_i gallons, $i = 1, \cdots, 4$ using a meter connected to his car's microcontroller. Denote the ground-truth fuel economy as k gallon/mile.

(a) Tom believes that the readings of the gas consumption Y_i are inaccurate but unbiased, so he set up a linear model $Y_i = kx_i + e_i$, $i = 1, \ldots, 4$, where e_i are measurement noise with zero-mean and variance σ^2. Express this model in the matrix-vector form. Explicitly define $\mathbf{y}$, $\mathbf{X}$, β, and $\mathbf{e}$.

(b) Use the normal equation, prove that least-squares estimator $\hat{k}$ for the fuel economy is

$$\hat{k} = \left(\sum_{i=1}^{4} x_i Y_i\right) \Bigg/ \left(\sum_{i=1}^{4} x_i^2\right).$$

(c) Tom's brother proposed another way to estimate the fuel economy

$$\tilde{k} = \left(\sum_{i=1}^{4} Y_i\right) \Bigg/ \left(\sum_{i=1}^{4} x_i\right).$$

With some manipulation, one can obtain that the variance of the least-squares estimator and Tom brother's estimator are respectively

$$\mathrm{Var}(\hat{k}) = \frac{\sigma^2}{\sum_{i=1}^{4} x_i^2} \quad \text{and} \quad \mathrm{Var}(\tilde{k}) = \frac{\sigma^2}{\frac{1}{4}\left(\sum_{i=1}^{4} x_i\right)^2}.$$

Tom plans to drive 1, 2, 2, and 3 miles for each test drive, respectively. Compare numerically the variance of the two estimators. Is the least-squares estimator better than the one proposed by Tom's brother? Give you justification. (The calculation should be done by hand and without using a calculator.)

Problem 3 (25 pts) [DC Power Supply] One technique for building a DC power supply is to take an AC signal and full-wave rectify it. That is, we put the AC signal $x(t)$ through a system that produces $y(t) = |x(t)|$ as its output.

a) Sketch the input and output waveforms if $x(t) = \sin(t)$. What are the fundamental periods of the input and the output?

b) If $x(t) = \sin(t)$, determine the coefficients of the Fourier series for the output $y(t)$.

c) What is the amplitude of the DC component of the input signal?

d) What is the amplitude of the DC component of the output signal?

Problem 4 (25 pts) [CTFT] Compute the inverse CTFT for the following signals. You must calculate the results using both i) the direct evaluation method based on the definition of the inverse CTFT and ii) the table of Fourier transform properties.

a) $2\delta(\omega + 2) - 2\delta(\omega - 2) + j\delta(\omega + 1) + j\delta(\omega - 1)$

b) $\mathrm{rect}(2\omega - 1)$

Problem 5 (25 pts) [Windowed Time-Domain Signals] Find and plot the Fourier transform of the following windowed sinusoidal signals.

$$x(t) = \begin{cases} \sin(10t), & -8 \le t \le 12, \\ 0, & \text{elsewhere.} \end{cases}$$