ECE 301 (Section 001) Homework 6
Spring 2022, Dr. Chau-Wai Wong
TA in Charge: Gavin Carter

Problem 1 (Properties of LTI Systems) Determine whether the following LTI systems are causal, memoryless, and stable, respectively. Justify your answers.

a) $h(t) = e^{-2t}u(t) + e^{t/100}u(t-1)$

b) $h[n] = (n+1)2^n u[n]$

c) $h[n] = 4^n u[2-n]$

d) $h(t) = e^{-2t}\left[u(t) - u(t-1)\right]$.

Problem 2 (Vector and Matrix Refresh) Seven data points are arranged as columns of a data matrix $\mathbf{X}$ given as follows:

$$\mathbf{X} = \begin{bmatrix} 2 & 1 & 0 & 0 & -1 & 0 & -2 \\ 2 & 0 & 1 & 0 & 0 & -1 & -2 \end{bmatrix}.$$

a) Draw all data points on a 2D plane by hand. Properly label the two axes. Clearly mark important axis tick values to facilitate a precise graphical description of the data.

b) Consider each point as a vector. Calculate the angle (in °) between $[2\ 2]^T$ and the five other points (excluding $[0\ 0]^T$), respectively, using the inner/dot product formula that involves the angle. Note that the angle between two vectors can be negative.

c) Calculate the matrix outer product for $\mathbf{X}$, namely, $\mathbf{R} = \mathbf{X}\mathbf{X}^T$. Show the intermediate steps of calculating each element of the 2-by-2 matrix $\mathbf{R}$.

d) The matrix outer product can also be calculated via $\mathbf{R} = \sum_{i=1}^{7} \mathbf{x}_i \mathbf{x}_i^T$, where $\mathbf{x}_i$ is the ith column of $\mathbf{X}$. Evaluate the numerical result. Note that each $\mathbf{x}_i \mathbf{x}_i^T$ is a 2-by-2 matrix.

e) Now, consider each column of $\mathbf{X}$ as a *single* block-entry of the matrix. Rewrite $\mathbf{X}$ and $\mathbf{X}^T$ into the form of the vector of blocks-entries, respectively. Use the vector multiplication rule to show that $\mathbf{R} = \sum_{i=1}^{7} \mathbf{x}_i \mathbf{x}_i^T$.

Problem 3 (Linearly independence, Basis, and Vector Space)

a) Are vectors $[1\ \ 2]$, $[4\ \ 5]$, and $[7\ \ 8]$ linearly independent? What about $[1\ \ 2\ \ 0]$, $[1\ \ -1\ \ 1]$, and $[0\ \ 0\ \ 1]$? Justify your answers.

b) You are given a vector space $V = \text{span}\left\{[-1\ \ 0\ \ 0], [0\ \ -1\ \ 0]\right\}$.

(i) Express V in a set representation.

(ii) Can you find a basis for V?

(iii) Are $[5\ \ 8\ \ 0]$, $[8\ \ 0\ \ 5]$, and $[0\ \ 5\ \ 8]$ in vector space V? Is yes, what are the coefficient for each vector of the basis you found in (ii)?

(iv) Draw all points of (iii) in a 3D coordinate. Illustrate vector space V using a plane formed by the vectors of the basis.

c) (Bonus, 5') Let

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & -1 & 0 \\ -1 & -1 & 2 & -3 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

What is the dimension of the column vector space of $\mathbf{A}$? What is the rank of $\mathbf{A}$?

Problem 4 (Projection) (This problem is now reassigned as Problem 1 of HW7. If you have already finished this problem, please submit it as Problem 1 of HW7.) You are given a data matrix

$$\mathbf{X} = \begin{bmatrix} -0.92 & 1.09 & -1.35 & 2.06 & -0.60 & -0.28 \\ -0.08 & 1.15 & -1.67 & 1.08 & -1.14 & 0.66 \end{bmatrix}.$$

and its principal component $\mathbf{u} = [0.7474 \;\; 0.6644]^T$.

a) Plot the data points using circles and the principal component using a line segment with one end at $[0, 0]^T$ on a 2D plane using Matlab.

b) Calculate the projection c_i for each data point $\mathbf{x}_i$ to $\mathbf{u}$. Calculate the sample variance of $\{c_i\}_{i=1}^n$, namely,

$$\widehat{\mathrm{Var}}\left(\{c_i\}\right) = \frac{1}{n-1} \sum_{i=1}^{n} \left(c_i - \bar{c}\right)^2,$$

where $\bar{c} \overset{\text{def}}{=} \frac{1}{n} \sum_{i=1}^{n} c_i$ is the sample mean.

c) Repeat b) using another unit vector $\mathbf{u}_a = [0.8 \;\; 0.6]^T$. Is the newer sample variance smaller than that in b)?

Problem 5 (Machine Learning Intro Video) Watch this 10-minute video:

`https://youtu.be/YH1gdd8kJXo`

Write 6–8 sentences to concisely summarize machine learning and/or artificial intelligence.

Group Study (1', bonus) Zoom: Take a screenshot of the whole team with everyone's webcam capturing his/her face. One of you will share the screen showing the specific homework assignment sheet that you are working on. In-Person: Take a selfie with all group members' faces in the photo. Capture the homework assignment sheet in the photo.

Include the screenshot/selfie in your own homework submission as the last "problem." Your screenshot/selfie gets you 1 bonus point; your group members need to do it separately to earn their bonus points.