ECE 301 (Section 001) Homework 8
Spring 2022, Dr. Chau-Wai Wong
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Problem 1 (Orthogonal Projection) Consider the set of inconsistent linear equations $\mathbf{Ax} = \mathbf{b}$ given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

a) Find the least-squares solution to these equations.

b) Find the "hat" matrix $\mathbf{H}$. Using Matlab, numerically verify $\mathbf{H} = \mathbf{H}^2$. Argue why.

c) Find the best approximation $\hat{\mathbf{b}} = \mathbf{Hb}$ to $\mathbf{b}$. Find the vector $\mathbf{b}' = (\mathbf{I} - \mathbf{H})\mathbf{b}$ and show numerically that it is orthogonal to $\hat{\mathbf{b}}$.

d) What does the matrix $\mathbf{I} - \mathbf{H}$ represent? If $\mathbf{H}$ is called the "orthogonal projector," can you think of a name for $\mathbf{I} - \mathbf{H}$? Numerically verify $\mathbf{I} - \mathbf{H} = (\mathbf{I} - \mathbf{H})^2$. Argue why.

e) In a 3-dimensional coordinate system, draw the column vectors of matrix $\mathbf{A}$, the column vector space of $\mathbf{A}$, $\mathbf{b}$, $\hat{\mathbf{b}}$, and $\mathbf{b}'$. Make sure that the drawing is reasonably accurate which can reflect the relationship among these quantities.

Problem 2 (Linear Regression in Matrix–Vector Form) An ECE student John is doing an electronic circuits lab during which he needs to determine the conductance of a resistor using a voltage meter, a current meter, and a DC power source. The voltage meter is connected in parallel with the resistor and the current meter is in series with the resistor. Both meters are analog devices so the readings recorded by John have errors. The power source is tunable and has a range of 1 to 5 V. Each time John will try a uniformly random input voltage level and record the readings of both voltage and current meters. Denote the voltage reading as x_i and the current reading as y_i for the ith measurement. Assume the true conductance $G = 2$ mΩ^{-1}.

a) Using a linear model $y_i = Gx_i + e_i$, where e_i is a zero-mean noise with standard deviation $\sigma_e = 0.1$ mA, simulate a dataset of $n = 10$ measurements. [Hint: You may use `rand()` to generate a uniformly random value in $[0, 1]$ and `0.1*randn()` to generate a zero-mean Gaussian noise with standard deviation 0.1. Throughout this problem, you may ignore the units such as mΩ^{-1} and mA and focus only on the numbers.]

b) Express the linear model in a matrix-vector form. Clearly indicate the matrices/vectors $\mathbf{y}, \mathbf{X}, \boldsymbol{\beta}$, and $\mathbf{e}$. Directly implement the formula of the least-squares (LS) estimator, $\hat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$, into a computer function that takes as two input vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, and output a number $\hat{G}$. Apply your function to the simulated data. What is the value of $\hat{G}$? [Hint: $\mathbf{X}$ is a n-by-1 "matrix," and $\boldsymbol{\beta}$ is a 1-by-1 "vector."]

c) John's friend, Tom, proposed a more intuitive estimator for the conductance: $\tilde{G} = \frac{1}{n}\sum_{i=1}^{n} \frac{y_i}{x_i}$. Let's call it "Tom's estimator" for convenience. Write a computer function that takes as two input vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, and output a number $\tilde{G}$. Apply your function to the simulated data. What is the value of $\tilde{G}$?

d) Generate 1,000 datasets. Repeatedly apply function written in (b) and collect 1,000 LS estimates and calculate the sample variance of these 1,000 values.

e) Use the 1,000 datasets generated in (d). Repeatedly apply function written in (c) and collect 1,000 Tom's estimates and calculate the sample variance of these 1,000 values. You should find that the LS estimator has a smaller variance than Tom's estimator.

Problem 3 (Deep Learning with Matlab) In recent updates, Matlab has put together well-guided tutorials for deep learning. This is one set of tutorials on "Deep Learning with Images": `https://www.mathworks.com/help/deeplearning/deep-learning-with-images.html`

Complete the following tutorials by running the example code:

a) "Create Simple Deep Learning Network for Classification."

b) (bonus, 10 pts) "Transfer Learning with Deep Network Designer."

Write a concise report consisting of key source code, images, and your explanations.

For more tutorials, see the left menu on this page: `https://www.mathworks.com/help/deeplearning/getting-started-with-deep-learning-toolbox.html`

Problem 4 (Eigen-signals/functions of an LTI System, bonus 5') On slide 21 of lecture 18, you were told that $e^{j\omega_0 t}$ is an eigen-signal of an LTI system $h(t)$. In other words, when the LTI system operates on an input signal $x(t) = e^{j\omega_0 t}$ [or $x(t)$ is sent into the LTI system], the output $y(t)$ is merely a scaled version of $x(t)$ for all $t \in \mathbb{R}$. Show that the scaling factor is

$$H(j\omega_0) = \int_{-\infty}^{\infty} h(t)e^{-j\omega_0 t}\, dt. \tag{1}$$

Recall that the input–output relation of an LTI system is related by $y(t) = h(t) * x(t)$.

Group Study (1', bonus) Zoom: Take a screenshot of the whole team with everyone's webcam capturing his/her face. One of you will share the screen showing the specific homework assignment sheet that you are working on. In-Person: Take a selfie with all group members' faces in the photo. Capture the homework assignment sheet in the photo.

Include the screenshot/selfie in your own homework submission as the last "problem." Your screenshot/selfie gets you 1 bonus point; your group members need to do it separately to earn their bonus points.