ECE 301 (001) Linear Systems
Spring 2023 Final Exam
Instructor: Dr. Chau-Wai Wong

This is a closed-book, closed-lecture notes exam. One single-sided, letter-sized, handwritten cheat sheet for the material covered in the third part of the course is allowed, and you may additionally bring your own cheat sheets for Midterm 1 and Midterm 2. Calculators are not allowed. You are required to answer ALL six (6) problems. Submission instructions:

1. Once the instructor says "stop writing," please use your mobile phone to "scan" every answer sheet. (After scanning, you must turn in the physical copy of your answer sheets to the instructor, or you will receive a zero for this exam.)
2. Do not forget to scan those pages on the back if you wrote answers on them. Double-check the scanned pages one by one on your phone to make sure they are properly captured.
3. You have a time window of 24 hours after the exam to submit your answers to Gradescope. Upload your scanned pages as a PDF file to Gradescope and tag each page.
4. There will be a 10-point penalty for every late hour after the submission deadline.
5. Turn in the physical copy of your answer sheets to the instructor in the front of the classroom.

Problem 1 (18 pts) [Geometric Series]

(a) (8') Without the help of the geometric series formula, prove that

$$S_1 = \sum_{n=a}^{b} r^n = \frac{r^a - r^{b+1}}{1-r},$$

where $a, b \in \mathbb{Z}$, $a < b$, and $|r| < 1$. Show intermediate steps clearly.

(b) (10') Derive the analytic form of

$$S_2 = \sum_{n=a}^{b} n r^n.$$

The final expression must not contain the summation sign.

Problem 2 (16 pts) [Convolution and Correlation]

You are given $x(t) = 2^{-\pi t} u(t-3)$ and $h(t) = u(t+3)$ for $t \in \mathbb{R}$. Using a graphical approach to determine how the interval $t \in \mathbb{R}$ should be divided into subintervals on which $y(t)$ will have different analytic expressions. Explicitly ***specify the range of each subinterval***. For each subinterval, set up an integral with ***proper lower and upper integration limits***, but do NOT proceed to evaluate the numerical result of the integral. For sanity check, the union of all subintervals must be $\mathbb{R}$.

(a) (8') When $y(t)$ is the *covolution* between $x(t)$ and $h(t)$:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.$$

(b) (8') When $y(t)$ is the *correlation* between $x(t)$ and $h(t)$:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(\tau-t)\,d\tau.$$

Problem 3 (16 pts) [Properties of LTI Systems] Determine whether the following LTI systems are causal, memoryless, and stable, respectively. Justify your answers.

(a) (8') $h[n] = e^n u[1-n]$

(b) (8') $h(t) = 3^{-t}\left[u(t) - u(t-2)\right]$.

Problem 4 (16 pts) [Fourier Transform] Compute Fourier transforms for the following signals. You must do the complete computation without relying on the tables.

(a) (5') $x(t) = e^{-|t-2|/5}$

(b) (5') $x(t) = \text{rect}\left(\frac{t}{3} - 1\right)$ (Final result must be represented in form of a sinc function.)

(c) (6') $x(t) = t - 1$ for $t \in [0, 1]$ otherwise $x(t) = 0$

Problem 5 (16 pts) [Windowed Time-Domain Signals] Find the real and imaginary parts of the Fourier transform of the following windowed sinusoidal signals.

$$x(t) = \begin{cases} \cos(10t), & -10 \le t \le 0. \\ 0, & \text{elsewhere.} \end{cases}$$

Problem 6 (18 pts) [Laplace Transform]

(a) (9') Calculate the Laplace transform for the following signal ***using the definition***:

$$x(t) = e^{3t}u(-t) + e^{2t}\sin(3t)u(-t).$$

Draw the associated region of convergence and the poles. (You don't need to draw the zeros.) Your answer must contain integration steps and the conditions for integrals to converge. Using transform pairs from the Laplace table is NOT allowed.

(b) (9') Determine the inverse Laplace transform ***with the help of the Laplace table***:

$$X(s) = -\frac{s+5}{s^2+4s+3}, \quad -3 < \mathcal{R}e\{s\} < -1.$$

CTFT pairs:

	Time domain $x(t)$	Frequency domain $X(j\omega)$
Delta	$\delta(t)$	1
Constant	$\frac{1}{2\pi}$	$\delta(\omega)$
Complex sinusoid	$\frac{e^{j\omega_0 t}}{2\pi}$	$\delta(\omega-\omega_0)$
Causal exponential	$e^{-at}u(t)$ $\text{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
Cosine	$\cos\omega_0 t$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
Sine	$\sin\omega_0 t$	$\pi j(\delta(\omega+\omega_0)-\delta(\omega-\omega_0))$
Periodic signal w/ period T	$x(t)$	$2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega-k\omega_0)$
Rectangle	$\text{rect}(t)$	$\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}=\text{sinc}\left(\frac{\omega}{2\pi}\right)$
Scaled rectangle	$\text{rect}\left(\frac{t}{2T_1}\right)$	$2T_1\frac{\sin\omega T_1}{\omega T_1}=2T_1\text{sinc}\left(\frac{\omega T_1}{\pi}\right)$
Sinc	$\text{sinc}(t)$	$\text{rect}\left(\frac{\omega}{2\pi}\right)$
Scaled sinc	$\frac{\lvert B\rvert}{2\pi}\text{sinc}\left(\frac{Bt}{2\pi}\right)$	$\text{rect}\left(\frac{\omega}{B}\right)$

CTFT properties:

Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Differentiation	$\frac{dx}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(j\omega)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Frequency scaling	$\frac{1}{\lvert b\rvert}x\left(\frac{t}{b}\right)$	$X(jb\omega)$
Frequency shifting	$x(t)e^{j\omega_0 t}$	$X(j(\omega-\omega_0))$
Parseval's theorem	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(j\omega)\rvert^2d\omega$	

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos \omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$