ECE 301 (Section 001) Homework 5
Spring 2023, Dr. Chau-Wai Wong
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Problem 1 (Time-Invariance & Linearity Properties)

a) Examine the time-invariance and linearity properties of the systems specified in Problem 3 of the previous HW.

b) Examine the time-invariance and linearity properties of the systems specified in Problem 4 of the previous HW. [Bonus 4' for part d)]

Problem 2 (Discrete Convolution Basics) Given signal $w[0] = -1$, $w[1] = 2$, $w[2] = -1$, and $w[n] = 0$, elsewhere, and signal $v[0] = 1$, $v[1] = 2$, $v[2] = 3$, $v[3] = 4$, and $v[n] = 0$, elsewhere.

a) Express $w[n]$ and $v[n]$ in the self-referenced form. Plot them in two separate figures. Remember to labels the axes.

b) Decompose input signal $x[n] = w[n]$ into three subsignals, $x_i[n]$ for $i = 1, 2, 3$, send them into an LTI system with impulse response $h[n] = v[n]$, and calculate outputs $y_i[n]$ for $i = 1, 2, 3$ and subsequently $y[n]$ using the graphical approach (see Convolution Example 1 from the slide deck of Chapter 2). Clearly define your choices of $x_i[n]$ signals. Explain how $y_i[n]$ signals are obtained.

c) Recalculate the output $y[n]$ by using the definition of convolution.

d) Repeat part b) by letting $x[n] = v[n]$ and $h[n] = w[n]$. Are we getting the same result as in b)?

Problem 3 (Discrete Convolution)

a) Use the definition of convolution, prove that for two generic signals $x[n]$ and $h[n]$, $x[n] * h[n] = h[n] * x[n]$. (You need to clearly demonstrate how a change of variables is done, and how the summation interval of the new dummy variable can be obtained from that of the old dummy variable.) How can this mathematical result be used to explain the finding in Problem 2d?

b) Compute the convolution $y[n] = x[n] * h[n]$ for $x[n] = \alpha^n u[n]$ and $h[n] = \beta^n u[n]$. Consider the cases $\alpha \neq \beta$ and $\alpha = \beta$.

c) Compute the convolution $y[n] = x[n] * h[n]$ for $x[n] = \left(-\frac{1}{2}\right)^n u[n-1]$ and $h[n] = 3^n u[1-n]$.

d) Let $h[n] = \sum_{k=-\infty}^{\infty} \delta[n-2k]$ be the impulse response of a discrete-time LTI system with input $x[n]$ given by:

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 2, \\ 0, & \text{elsewhere.} \end{cases} \tag{1}$$

(i) Find the output $y[n]$ of the system.

(ii) Is $y[n]$ periodic? If yes, find its fundamental period.

Problem 4 (The Human Body as an "LTI System") The concentration of an injected (or orally ingested) drug in the blood is commonly modeled as a decaying exponential. This means that if a human body is injected with a drug to time 0, its concentration after n hours is given by $h[n] = \alpha^n u[n]$, where α is defined by the half-life $n_0 > 0$, meaning that $h[n_0] = 0.5h[0]$. You can think of a drug injection as an impulse $x_0[n] = \delta[n]$ and $h[n]$ as the impulse response of the human body, which metabolizes the drug over time.

a) Given that the half-life of a certain drug is 8 hours, what is $h[n]$? Also, find $y[n] = h[n] * x_0[n]$.

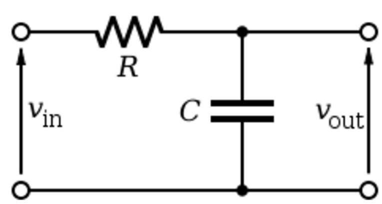
b) If instead of an injection, the drug is administered through a steady intravenous (IV) drip starting at time 0, what will be the concentration of the drug as a function of time? Specifically, if $x_1[n] = 0.2u[n]$ find $y[n] = x_1[n] * h[n]$.

c) The drip procedure is not producing a sufficient response. So, a strong extra dose of the drug is injected after 8 hours into the dripping process, namely $x_2[n] = 2\delta[n-8]$. Find the new concentration $y[n]$ as a function of time.

d) Compare the concentrations of the drug in the blood after 16 hours under the three different scenarios, (i) injection at time 0, (ii) drip starting at time 0, (iii) drip starting at time 0 and then injection as modeled by $x_2[n]$.

Problem 5 (Bonus, 20') (Delta Function as a Limiting Function)

a) (15') The RC circuit below has supply voltage $v_{\text{in}}(t)$ and capacitor voltage $v_{\text{out}}(t)$. We consider the supply voltage as input of the system, i.e., $x(t) = v_{\text{in}}(t)$ and capacitor voltage as the output, i.e., $y(t) = v_{\text{out}}(t)$. The output is $y(t) = \left[1 - e^{-5(t-t_0)}\right] u(t - t_0)$ when supplied with a unit-step input voltage $x(t) = u(t - t_0)$.



(i) Without going into mathematical derivation/manipulation, argue with intuition why the RC circuit can be viewed as a time-invariant system.

(ii) Suppose the input voltage is 1 during the interval of $t = 0$ to 0.5 seconds and 0 otherwise, i.e., $x(t) = u(t) - u(t - 0.5)$. Using the superposition principle you learned from the circuit theory (note that it is equivalent to the superposition property in this course), show that the analytic form of the output voltage is

$$y(t) = \begin{cases} 1 - e^{-5t}, & 0 \leq t \leq 0.5, \\ 11.18e^{-5t}, & t > 0.5, \\ 0, & \text{else.} \end{cases}$$

(iii) How is the superposition principle related to the linearity property of the system of the RC circuit?

(iv) What are the analytic forms of the output voltage given the following input voltage?

(1) $x(t) = 2[u(t) - u(t - 0.5)]$

(2) $x(t) = 4[u(t) - u(t - 0.25)]$

(3) $x(t) = 20[u(t) - u(t - 0.05)]$

(v) Using the same amplitude scale and time scale on the same Matlab plot, visualize the output voltages $y(t)$ obtained in (iv) over $0 \leq t \leq 1$. Note that, using (iv)(1) as an example, you can either draw a piecewise function separately for `t = 0 : 0.01 : 0.5` and `t = 0.5 : 0.01 : 1` or in one piece for `t = 0 : 0.01 : 1`.

b) (5') You should notice that the sequence of curves $y(t)$ you plotted for a)(iv) are converging. Mathematically prove that the limiting curve is the delta function. [Hint: Guess a common expression for all $x(t)$'s in a)(iv) and parameterize the expression by a variable. Drive the variable to an extreme value and see the impact to $y(t)$.]

Group Study (1', bonus) Zoom: Take a screenshot of the whole team with everyone's webcam capturing his/her face. One of you will share the screen showing the specific homework assignment sheet that you are working on. In-Person: Take a selfie with all group members' faces in the photo. Capture the homework assignment sheet in the photo.

Include the screenshot/selfie in your own homework submission as the last "problem." Your screenshot/selfie gets you 1 bonus point; your group members need to do it separately to earn their bonus points.