ECE 301 (001) Linear Systems
Spring 2023 Midterm Exam 2
Instructor: Dr. Chau-Wai Wong

This is a closed-book, closed-lecture notes exam. One single-sided, letter-sized, handwritten cheat sheet for the material covered in Midterm 2 is allowed, and you may additionally bring your own cheat sheet for Midterm 1. Calculators are not allowed. You are required to answer ALL four (4) problems. Submission instructions:

1. Once the instructor says "stop writing," please use your mobile phone to "scan" every answer sheet. (After scanning, you must turn in the physical copy of your answer sheets to the instructor, or you will receive zero for this exam.)

2. Do not forget to scan those pages on the back if you wrote answers on them. Double check the scanned pages one by one on your phone to make sure they are properly captured.

3. You have a time window of 6 hours after the exam to submit your answers to Gradescope. Upload your scanned pages as a PDF file to Gradescope and tag each page.

4. There will be a 10-point penalty for every late hour after the submission deadline.

5. Turn in the physical copy of your answer sheets to the instructor in the front of the classroom.

Problem 1 (25 pts) [Properties of LTI Systems] Determine whether the following LTI systems are causal, memoryless, and stable, respectively. Justify your answers.

(a) $h(t) = e^{\pi t}u(t) + e^{-\pi t}u(t-1)$

(b) $h[n] = (n-2)\, e^{n} u[n]$

Problem 2 (25 pts) [Basis and Vector Space] You are given a vector space $V = \text{span}\,\{[1,-2,0],[-1,0,3]\}$.

(a) Express V in a set representation.

(b) Can you find a basis for V?

(c) Are $[1,1,1]$, $[0,6,-9]$, and $[3,2,1]$ in vector space V? If yes, what are the coefficient for each vector of the basis you found in (b)?

(d) Illustrate vector space V using a plane formed by the vectors of the basis.

Problem 3 (25 pts) [Linear Regression in Matrix–Vector Form] An online game player, Durse, plans to test how well they can do level grinding (i.e., gaining their experience points for raising the level of their game character) per one minute's play of the game. They will conduct 4 tests of x_i hours each, $i = 1, \ldots, 4$. At the beginning of each test, a new gaming account will be created. At the end of the test, Durse will read from the screen the total number of earned experience points Y_i, $i = 1, \ldots, 4$. Denote the ground-truth point-earning rate as k points/hour.

(a) Durse believes that the total number of the earned experience points Y_i they read from the experience bar displayed on the screen is somewhat inaccurate but unbiased, so they set up a linear model $Y_i = kx_i + e_i$, $i = 1, \ldots, 4$, where e_i are measurement noise with zero-mean and variance σ^2. Express this model in the matrix–vector form. Explicitly define $\mathbf{y}$, $\mathbf{X}$, $\boldsymbol{\beta}$, and $\mathbf{e}$.

(b) Use the normal equation, prove that least-squares estimator $\hat{k}$ for the point-earning rate is:

$$\hat{k} = \left(\sum_{i=1}^{4} x_i Y_i\right) \Bigg/ \left(\sum_{i=1}^{4} x_i^2\right).$$

(c) Durse's friend, Dhase, proposed another way to estimate the point-earning rate as follows:

$$\tilde{k} = \left(\sum_{i=1}^{4} Y_i\right) \Bigg/ \left(\sum_{i=1}^{4} x_i\right).$$

With some manipulations, one can obtain that the variance of the least-squares estimator and Dhase's estimator are respectively

$$\mathrm{Var}(\hat{k}) = \frac{\sigma^2}{\sum_{i=1}^{4} x_i^2} \quad \text{and} \quad \mathrm{Var}(\tilde{k}) = \frac{\sigma^2}{\frac{1}{4}\left(\sum_{i=1}^{4} x_i\right)^2}.$$

Durse plans to test how well they can do level grinding by repeatedly playing the game for 3, 2, 2, and 1 hours, respectively. Compare numerically the variance of the two estimators. Is the least-squares estimator better than the one proposed by Dhase? Give your justification. (The calculation should be done by hand and without using a calculator.)

Problem 4 (25 pts) [Inverse Fourier Transform] Compute the inverse CTFT for the following signals. You must calculate the results using both i) the direct evaluation method based on the definition of the inverse CTFT and ii) the table of Fourier transform properties.

(a) $2\delta(\omega + 2) + 2\delta(\omega - 2) - j\delta(\omega + 1) + j\delta(\omega - 1)$

(b) $\mathrm{rect}(\frac{\omega}{2} - 1)$

CTFT pairs:

	Time domain *x(t)*	Frequency domain *X(j ω)*
Delta	$\delta(t)$	1
Constant	$\frac{1}{2\pi}$	$\delta(\omega)$
Complex sinusoid	$\frac{e^{j\omega_0 t}}{2\pi}$	$\delta(\omega-\omega_0)$
Causal exponential	$e^{-at}u(t)$ $\mathrm{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
Cosine	$\cos\omega_0 t$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
Sine	$\sin\omega_0 t$	$\pi j(\delta(\omega+\omega_0)-\delta(\omega-\omega_0))$
Periodic signal w/ period T	$x(t)$	$2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega-k\omega_0)$
Rectangle	$\mathrm{rect}(t)$	$\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}=\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$
Scaled rectangle	$\mathrm{rect}\left(\frac{t}{2T_1}\right)$	$2T_1\frac{\sin\omega T_1}{\omega T_1}=2T_1\mathrm{sinc}\left(\frac{\omega T_1}{\pi}\right)$
Sinc	$\mathrm{sinc}(t)$	$\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$
Scaled sinc	$\frac{\lvert B\rvert}{2\pi}\mathrm{sinc}\left(\frac{Bt}{2\pi}\right)$	$\mathrm{rect}\left(\frac{\omega}{B}\right)$

CTFT properties:

Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Differentiation	$\frac{dx}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(j\omega)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Frequency scaling	$\frac{1}{\lvert b\rvert}x\left(\frac{t}{b}\right)$	$X(jb\omega)$
Frequency shifting	$x(t)e^{j\omega_0 t}$	$X(j(\omega-\omega_0))$
Parseval's theorem	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(j\omega)\rvert^2d\omega$	