ECE 301 (Section 001) Homework 7
Spring 2024, Dr. Chau-Wai Wong
TA in Charge: Jordan Zhang

Problem 1 (Properties of LTI Systems) Determine whether the following LTI systems are causal, memoryless, and stable, respectively. Justify your answers.

- **a)** $h(t) = e^{-2t}u(t) + e^{t/100}u(t-1)$
- **b)** $h[n] = (n+1)2^n u[n]$
- **c)** $h[n] = 4^n u[2-n]$
- **d)** $h(t) = e^{-2t}\left[u(t) - u(t-1)\right].$

Problem 2 (Vector and Matrix Refresh) Seven data points are arranged as columns of a data matrix $\mathbf{X}$ given as follows:

$$\mathbf{X} = \begin{bmatrix} 2 & 1 & 0 & 0 & -1 & 0 & -2 \\ 2 & 0 & 1 & 0 & 0 & -1 & -2 \end{bmatrix}.$$

- **a)** Draw all data points on a 2D plane by hand. Properly label the two axes. Clearly mark important axis tick values to facilitate a precise graphical description of the data.
- **b)** Consider each point as a vector. Calculate the angle (in °) between $[2\ 2]^T$ and the five other points (excluding $[0\ 0]^T$), respectively, using the inner/dot product formula that involves the angle. Note that the angle between two vectors can be negative.
- **c)** Calculate the matrix outer product for $\mathbf{X}$, namely, $\mathbf{R} = \mathbf{X}\mathbf{X}^T$. Show the intermediate steps of calculating each element of the 2-by-2 matrix $\mathbf{R}$.
- **d)** The matrix outer product can also be calculated via $\mathbf{R} = \sum_{i=1}^{7} \mathbf{x}_i\mathbf{x}_i^T$, where $\mathbf{x}_i$ is the ith column of $\mathbf{X}$. Evaluate the numerical result. Note that each $\mathbf{x}_i\mathbf{x}_i^T$ is a 2-by-2 matrix.
- **e)** Now, consider each column of $\mathbf{X}$ as a *single* block-entry of the matrix. Rewrite $\mathbf{X}$ and $\mathbf{X}^T$ into the form of the vector of blocks-entries, respectively. Use the vector multiplication rule to show that $\mathbf{R} = \sum_{i=1}^{7} \mathbf{x}_i\mathbf{x}_i^T$.

Problem 3 (Linearly independence, Basis, and Vector Space)

- **a)** Are vectors $[1\ \ 2], [4\ \ 5]$, and $[7\ \ 8]$ linearly independent? What about $[1\ \ 2\ \ 0], [1\ \ -1\ \ 1]$, and $[0\ \ 0\ \ 1]$? Justify your answers.
- **b)** You are given a vector space $V = \text{span}\{[-1\ \ 0\ \ 0], [0\ \ -1\ \ 0]\}$.
  - (i) Express V in a set representation.
  - (ii) Can you find a basis for V?
  - (iii) Are $[5\ \ 8\ \ 0]$, $[8\ \ 0\ \ 5]$, and $[0\ \ 5\ \ 8]$ in vector space V? Is yes, what are the coefficient for each vector of the basis you found in (ii)?
  - (iv) Draw all points of (iii) in a 3D coordinate. Illustrate vector space V using a plane formed by the vectors of the basis.

c) (Bonus, 5') Let

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & -1 & 0 \\ -1 & -1 & 2 & -3 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

What is the dimension of the column vector space of $\mathbf{A}$? What is the rank of $\mathbf{A}$?

Problem 4 (Orthogonal and Unitary Matrices)

You have learned in class that a real-valued *orthogonal matrix* $\mathbf{P}$ has the property that $\mathbf{P}^T\mathbf{P} = \mathbf{P}\mathbf{P}^T = \mathbf{I}$. In signal processing, oftentimes signals are represented in complex values. We can define for a complex-valued square matrix a similar concept called the *unitary matrix*. A unitary matrix $\mathbf{Q}$ satisfies the property that $\mathbf{Q}^H\mathbf{Q} = \mathbf{Q}\mathbf{Q}^H = \mathbf{I}$, where "^{H}" is the Hermitian operator that is a combination of both the transpose, "^{T}," and the complex conjugate, "*."

a) The discrete Fourier transform (DFT) matrix for a signal of length 3 is defined as $\mathbf{Q}_3 = [q_{kn}]_{k,n\in\{0,1,2\}}$, where $q_{kn} = \frac{1}{\sqrt{3}}\exp(-j\frac{2\pi}{3}kn)$. Explicitly write out this 3-by-3 matrix. Simplify each entry but do not evaluate numerically the complex exponentials.

b) Verify that $\mathbf{Q}_3$ is a unitary matrix.

c) Compute the forward discrete Fourier transform for signal $\mathbf{x} = [1.01, 0.99, 0.97]^T$ to obtain a transformed signal $\mathbf{z} = \mathbf{Q}_3\,\mathbf{x}$, where $\mathbf{z} = [z_0, z_1, z_2]^T$. How large are the norms for z_0, z_1, and z_2?

d) Now, zero out z_2 to obtain a new vector $\mathbf{z}_{\text{compressed}}$. Compute the inverse transform using $\hat{\mathbf{x}} = \mathbf{Q}_3^H\,\mathbf{z}_{\text{compressed}}$. Is $\hat{\mathbf{x}}$ similar to $\mathbf{x}$? Can you guess why?

Problem 5 (Bonus 5') (Machine Learning Intro Video) Watch this 10-minute video:

https://youtu.be/YH1gdd8kJXo

Write 6–8 sentences to concisely summarize machine learning and/or artificial intelligence.

Group Study (1', bonus) In-Person: Take a selfie with all group members' faces in the photo. Capture in the photo the homework assignment sheet that you are working on. Zoom: Take a screenshot of the whole team with everyone's webcam capturing his/her face. One of you will share the screen showing the specific homework assignment sheet.

Include the screenshot/selfie in your own homework submission as the last "problem." Your screenshot/selfie gets you 1 bonus point; your group members need to do it separately to earn their bonus points.