ECE 301 (001) Linear Systems
Spring 2024 Midterm Exam 2
Instructor: Dr. Chau-Wai Wong

This is a closed-book, closed-lecture notes exam. One single-sided, letter-sized, handwritten cheat sheet for the material covered in Midterm 2 is allowed, and you may additionally bring your own cheat sheet for Midterm 1. Calculators are not allowed. You are required to answer ALL four (4) problems. Submission instructions for in-class students:

1. Once the instructor says "stop writing," please use your mobile phone to "scan" every answer sheet. (After scanning, you must turn in the physical copy of your answer sheets to the instructor, or you will receive zero for this exam.)
2. Do not forget to scan those pages on the back if you wrote answers on them. Double check the scanned pages one by one on your phone to make sure they are properly captured.
3. You have a time window of 6 hours after the exam to submit your answers to Gradescope. Upload your scanned pages as a PDF file to Gradescope and tag each page.
4. There will be a 5-point penalty for every late hour after the submission deadline.
5. Turn in the physical copy of your answer sheets to the instructor in the front of the classroom.

If you take the exam at a university testing center, the proctoring staff will scan your papers and send them to Dr. Wong. Dr. Wong will later forward the scanned PDF to you, and you will be responsible for uploading the PDF file to Gradescope and tagging each page within 24 hours of receiving Dr. Wong's email.

Problem 1 (25 pts) [Properties of LTI Systems] Determine whether the following LTI systems are causal, memoryless, and stable, respectively. Justify your answers.

(a) $h[n] = 2^n u[4-n]$

(b) $h(t) = 2^{-t}\left[u(t) - u(t-2)\right].$

Problem 2 (25 pts) [Basis and Vector Space] You are given a vector space $V = \text{span}\{[2,0,1],[0,-1,2]\}$.

(a) Express V in a set representation.

(b) Can you find a basis for V?

(c) Are $[2,1,1]$, $[2,1,-1]$, and $[2,-1,1]$ in vector space V? If yes, what are the coefficient for each vector of the basis you found in (b)?

(d) Illustrate vector space V using a plane formed by the vectors of the basis.

Problem 3 (25 pts) [Linear Regression in Matrix–Vector Form] An ECE student, Bob, plans to test the battery discharge rate of his iPad in terms of percentage drop per one minute's watch of YouTube. He will do four 4 tests of x_i minutes each, $i = 1, \ldots, 4$, and will read from the iPad's display the corresponding battery percentage drop in Y_i percentage, $i = 1, \ldots, 4$. Denote the ground-truth battery discharge rate as k percent/minute.

(a) Bob believes that the readings of the battery percentage drop Y_i is somewhat inaccurate but unbiased, so he set up a linear model $Y_i = kx_i + e_i$, $i = 1, \ldots, 4$, where e_i are measurement noise with zero-mean and variance σ^2. Express this model in the matrix–vector form. Explicitly define $\mathbf{y}$, $\mathbf{X}$, $\boldsymbol{\beta}$, and $\mathbf{e}$.

(b) Use the normal equation, prove that least-squares estimator $\hat{k}$ for the battery discharge rate

$$\hat{k} = \left(\sum_{i=1}^{4} x_i Y_i \right) \Bigg/ \left(\sum_{i=1}^{4} x_i^2 \right).$$

(c) Bob's friend proposed another way to estimate the battery discharge rate

$$\tilde{k} = \frac{1}{4} \sum_{i=1}^{4} \frac{Y_i}{x_i}.$$

With some manipulations, one can obtain that the variance of the least-squares estimator and Bob's friend's estimator are respectively

$$\mathrm{Var}(\hat{k}) = \frac{\sigma^2}{\sum_{i=1}^{4} x_i^2} \quad \text{and} \quad \mathrm{Var}(\tilde{k}) = \frac{\sigma^2}{16} \sum_{i=1}^{4} x_i^{-2}.$$

Bob plans to test the battery by watching YouTube for 3, 2, 2, and 1 minutes, respectively. Compare numerically the variance of the two estimators. Is the least-squares estimator better than the one proposed by Bob's friend? Give your justification. (The calculation should be done by hand and without using a calculator.)

Problem 4 (25 pts) [Inverse Fourier Transform] Compute the inverse CTFT for the following signals. You must calculate the results using both i) the direct evaluation method based on the definition of the inverse CTFT and ii) the table of Fourier transform properties.

(a) $j\delta(\omega + 2) + j\delta(\omega - 2) - 2\delta(\omega + 1) + 3\delta(\omega - 1)$

(b) $\mathrm{rect}(1 - 0.1\omega)$

Answer sheet for Problem [ ] Name: Obtained Points:

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Draft paper

Answer sheet for Problem [ ] Name: Obtained Points:

Answer sheet for Problem [ ] Name: Obtained Points:

CTFT pairs:

	Time domain $x(t)$	Frequency domain $X(j\omega)$
Delta	$\delta(t)$	1
Constant	$\frac{1}{2\pi}$	$\delta(\omega)$
Complex sinusoid	$\frac{e^{j\omega_0 t}}{2\pi}$	$\delta(\omega-\omega_0)$
Causal exponential	$e^{-at}u(t)$ $\mathrm{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
Cosine	$\cos\omega_0 t$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
Sine	$\sin\omega_0 t$	$\pi j(\delta(\omega+\omega_0)-\delta(\omega-\omega_0))$
Periodic signal w/ period T	$x(t)$	$2\pi\sum_{k=-\infty}^{\infty}a_k\delta(\omega-k\omega_0)$
Rectangle	$\mathrm{rect}(t)$	$\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}=\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$
Scaled rectangle	$\mathrm{rect}\left(\frac{t}{2T_1}\right)$	$2T_1\frac{\sin\omega T_1}{\omega T_1}=2T_1\mathrm{sinc}\left(\frac{\omega T_1}{\pi}\right)$
Sinc	$\mathrm{sinc}(t)$	$\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$
Scaled sinc	$\frac{\lvert B\rvert}{2\pi}\mathrm{sinc}\left(\frac{Bt}{2\pi}\right)$	$\mathrm{rect}\left(\frac{\omega}{B}\right)$

CTFT properties:

Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Differentiation	$\frac{dx}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(j\omega)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Frequency scaling	$\frac{1}{\lvert b\rvert}x\left(\frac{t}{b}\right)$	$X(jb\omega)$
Frequency shifting	$x(t)e^{j\omega_0 t}$	$X(j(\omega-\omega_0))$
Parseval's theorem	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(j\omega)\rvert^2d\omega$	