ECE 411 Homework 1 (Fall 2022)
Instructor: Dr. Chau-Wai Wong
Material Covered: Prerequisites, Machine Learning Overview

Problem 1 (20 points) [Course Prerequisite: Calculus]

(a) (10 points) [Partial Derivatives] Given constants Y_i, $i = 1, 2, 3$ and x_{ij}, $i = 1, 2, 3$, $j = 1, 2$, J is a function of independent variables β_1 and β_2 defined as follows:

$$J(\beta_1, \beta_2) = \Big(Y_1 - x_{11}\beta_1 - x_{12}\beta_2\Big)^2 + \Big(Y_2 - x_{21}\beta_1 - x_{22}\beta_2\Big)^2 + \Big(Y_3 - x_{31}\beta_1 - x_{32}\beta_2\Big)^2. \quad (1)$$

Under some technical conditions, we know that $J(\beta_1, \beta_2)$ to have a unique minimizer at $(\beta_1, \beta_2) = (\beta_1^*, \beta_2^*)$. With the help of partial differentiation you learned in calculus, prove that β_1^* and β_2^* satisfy the following relationships:

$$\sum_{i=1}^{3} Y_i x_{i1} = \sum_{i=1}^{3}\sum_{j=1}^{2} x_{ij}\beta_j^* x_{i1}, \quad (2a)$$

$$\sum_{i=1}^{3} Y_i x_{i2} = \sum_{i=1}^{3}\sum_{j=1}^{2} x_{ij}\beta_j^* x_{i2}. \quad (2b)$$

You will later learn in class that they are called *normal equations.*

(b) (10 points) [Limit] Simplify the following expression

$$\lim_{\beta\to\infty} \frac{\exp(\beta x)}{\exp(\beta x) + \exp(\beta y)} \quad (3)$$

(i) when $x > y$ and $x < y$, respectively. (ii) Explain the results in your own words and/or using graphs.

Problem 2 (20 points) [Course Prerequisite: Probability/Statistics and Matrix Operations]

(a) (10 points) [Linearity of the Expectation Operator] X is a random variable with probability density function $f(x)$, $x \in \mathbb{R}$. a and b are constants. (i) Prove using the definition of *expectation* that $\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b$. (For those of you who took ECE 301 Linear Systems, this is the *linearity property* of the expectation operator $\mathbb{E}[\cdot]$.) (ii) Explain intuitively or using a real-world example why the expectation is a linear operator.

(b) (10 points) [Covariance Matrix] The *variance–covariance matrix* or simply the *covariance matrix*, which will be used later in this course, can be regarded as a generalization from a measure of the variation of a single random variable X to a vector of random variables $\mathbf{X} = [X_1, \ldots, X_n]^T$. Its mathematical definition is as follows

$$\mathrm{Cov}(\mathbf{X}) = \mathbb{E}\left[(\mathbf{X} - \mathbb{E}[\mathbf{X}])(\mathbf{X} - \mathbb{E}[\mathbf{X}])^T\right] \quad (4)$$

and your job is to verify using simple matrix operations that the covariance matrix is a square matrix consisting of variance terms on the diagonal and covariance terms off the diagonal.

Prove that the elementwise expression of the covariance matrix for a length-3 random vector $\mathbf{X} = [X_1, X_2, X_3]^T$ is as follows:

$$\text{Cov}(\mathbf{X}) = \begin{bmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) & \text{Cov}(X_1, X_3) \\ \text{Cov}(X_2, X_1) & \text{Var}(X_2) & \text{Cov}(X_2, X_3) \\ \text{Cov}(X_3, X_1) & \text{Cov}(X_3, X_2) & \text{Var}(X_3) \end{bmatrix}. \tag{5}$$

Hint: Write variance and covariance in terms of $\mathbb{E}[\cdot]$ will be helpful.

Problem 3 [Overview of AI/ML] Continue to watch the video: *NOVA Wonders Can We Build a Brain?*

(a) (20 points) Write a concise summary for machine learning/artificial intelligence from the technical perspective. Elaborate the perspective using 3–5 sentences.

(b) (5 points, bonus) Write a concise summary for machine learning/artificial intelligence from the ethical perspective. Elaborate the perspective using 3–5 sentences.