

**ECE 411 Homework 8 (Fall 2022)**  
**Instructor: Dr. Chau-Wai Wong**  
**Material Covered: Maximum Likelihood, Generalized Linear Models**

**Problem 1** (20 points) [Stock Market Direction Prediction] Complete *ISLR-4.6.1-5, 4.7.10*. Be super concise when reporting the results for 4.6.1-5, and be selective when reporting the results for 4.7.10.

For Python users, please download *Smarket.csv data*, *Weekly.csv data* and follow the text book's instructions while referring to the "equivalence" Python code of *ISLR-4.6.1-5*.

**Problem 2** (20 points) [Maximum Likelihood Estimator (MLE)]

(a) Calculate the MLE for variance  $\theta$  (or  $\sigma^2$ , a more common notation if you prefer) for a random sample  $X_1, \dots, X_n$  drawn from the normal distribution with PDF shown as follows:

$$f(x; \theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-(x-\mu)^2/2\theta}, \quad -\infty < x < \infty. \quad (1)$$

Once you are done with partial differentiating against  $\theta$ , try to do partial derivative directly against  $\sigma$ . Leave your intermediate steps there and comment on where you have encountered difficulty.

(b) Calculate the MLE for parameter  $b$  for a random sample  $X_1, \dots, X_n$  drawn from an exponential distribution with PDF of the following form:

$$f(x; b) = \frac{1}{b} e^{-x/b}, \quad x \geq 0. \quad (2)$$

(c) (work on it after 10/31's lecture) The exponential distribution is more often parameterized using the rate parameter  $\lambda$  with PDF of the following form:

$$f(x; \lambda) = \lambda e^{-\lambda x}, \quad x \geq 0. \quad (3)$$

Use the invariance principle of MLE, show that  $\hat{\lambda}_{\text{MLE}} = 1/\bar{X}$ .

**Problem 3** (20 points) (work on it after 10/31's lecture) [Generalized Linear Model (GLM)] Response  $Y_i \sim B(n, p_i)$  is a binomial random variable in which  $n$  is known. The (conditional) PDF is shown as follows:

$$\mathbb{P}[Y_i = k | \underline{X}_i = \underline{x}_i] = \binom{n}{k} p_i^k (1 - p_i)^{n-k}, \quad k \in \{0, 1, \dots, n\}. \quad (4)$$

(a) Explain why the linear regression may not be the best fit to find the relation between  $Y_i$  and a set of predictors  $X_{i,1}, \dots, X_{i,q}$ .

(b) One proposes to link the conditional mean  $\mu_i$  and the predictors  $\underline{x}_i$  using a generalized linear model shown as follows:

$$g(\mu_i) = \underline{\beta}^T \underline{x}_i \quad (5)$$

where  $g(u) = \log\left(\frac{u}{n-u}\right)$  and  $\mu_i = \mathbb{E}[Y_i | \underline{X}_i = \underline{x}_i] = np_i$ . From the variable transformation viewpoint, show that  $g(\cdot)$  matches the ranges for the two sides of Eq. (5).

(c) Rewrite the PDF into an exponential family form shown as follows:

$$f_Y(y; \theta) = \exp\left(\frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi)\right), \quad (6)$$

where  $\theta$  is the natural parameter. Show that  $g(\cdot)$  in (b) is the canonical link function when taking  $\mu_i$  as the input.

*(You are given 3 required problems. The rest of time should be devoted to the term project.)*