ECE 411 Homework 8 (Fall 2022)
Instructor: Dr. Chau-Wai Wong
TA in Charge: Mr. Prasun Datta
Material Covered: Maximum Likelihood, Generalized Linear Models

Problem 1 (20 points) [Stock Market Direction Prediction] Complete *ISLR-4.6.1–5, 4.7.10.* Be super concise when reporting the results for 4.6.1–5, and be selective when reporting the results for 4.7.10.

For Python users, please download *Smarket.csv data*, *Weekly.csv data* and follow the text book's instructions while referring to *the "equivalence" Python code of ISLR-4.6.1–5*.

Problem 2 (20 points) [Maximum Likelihood Estimator (MLE)]

a) Calculate the MLE for variance θ (or σ^2, a more common notation if you prefer) for a random sample $X_1, \ldots, X_n$ drawn from the normal distribution with PDF shown as follows:

$$f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}}\, e^{-(x-\mu)^2/2\theta}, \quad -\infty < x < \infty. \tag{1}$$

Once you are done with partial differentiating against θ, try to do partial derivative directly against σ. Leave your intermediate steps there and comment on where you have encountered difficulty.

b) Calculate the MLE for parameter b for a random sample $X_1, \ldots, X_n$ drawn from an exponential distribution with PDF of the following form:

$$f(x;b) = \frac{1}{b}\, e^{-x/b}, \quad x \geq 0. \tag{2}$$

c) The exponential distribution is more often parameterized using the rate parameter λ with PDF of the following form:

$$f(x;\lambda) = \lambda\, e^{-\lambda x}, \quad x \geq 0. \tag{3}$$

Use the invariance principle of MLE, show that $\hat{\lambda}_{\text{MLE}} = 1/\bar{X}$.

Problem 3 (20 points) [Generalized Linear Model (GLM)] Response $Y_i \sim \mathrm{B}(n, p_i)$ is a binomial random variable in which n is known. The (conditional) PDF is shown as follows:

$$\mathbb{P}[Y_i = k | \underset{\sim}{X}_i = \underset{\sim}{x}_i] = \binom{n}{k} p_i^k (1-p_i)^{n-k}, \quad k \in \{0, 1, \ldots, n\}. \tag{4}$$

a) Explain why the linear regression may not the best fit to find the relation between Y_i and a set of predictors $X_{i,1}, \ldots, X_{i,q}$.

b) (work on it after 11/1's lecture) One proposes to link the conditional mean μ_i and the predictors $\underset{\sim}{x}_i$ using a generalized linear model shown as follows:

$$g(\mu_i) = \underset{\sim}{\beta}^T \underset{\sim}{x}_i \tag{5}$$

where $g(u) = \log(\frac{u}{n-u})$ and $\mu_i = \mathbb{E}[Y_i | \underset{\sim}{X}_i = \underset{\sim}{x}_i] = np_i$. From the variable transformation viewpoint, show that $g(\cdot)$ matches the ranges for the two sides of Eq. (5).

c) (work on it after 11/1's lecture) Rewrite the PDF into an exponential family form shown as follows:

$$f_Y(y;\theta) = \exp\left(\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right), \tag{6}$$

where θ is the natural parameter. Show that $g(\cdot)$ in (b) is the canonical link function when taking μ_i as the input.

(You are given 3 required problems. The rest of time should be devoted to the term project.)