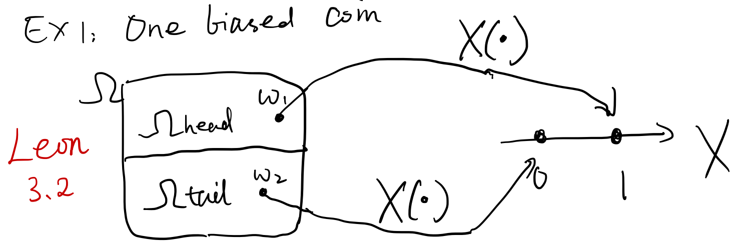
Review of Prob basics

Random Variable: A mapping for sample space Ω / S to real line $\mathbb{R}$.

Ex 1: One biased coin



$w_1, w_2 \in \Omega = \Omega_{head} \cup \Omega_{tail}$

$w_1 \in \Omega_{head}$, $w_2 \in \Omega_{tail}$

$X(w_1) = 1$, $X(w_2) = 0$

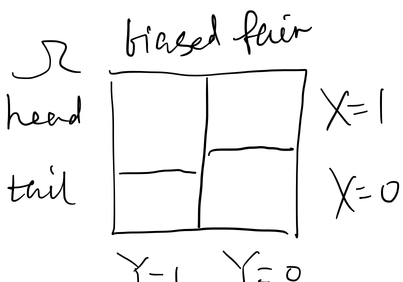
$$\underbrace{\mathbb{P}[X=0]}_{\text{short hand}} \overset{\text{def}}{=} \mathbb{P}[\{w : X(w) = 0, w \in \Omega\}]$$

Say: $\mathbb{P}[X=0] = 0.45$, $\mathbb{P}[X=1] = 0.55$.

Mean $\boxed{\mathbb{E}[X] = \sum_i x_i P_X(x_i)} = 0 \times 0.45 + 1 \times 0.55 = 0.55$

3.3, 4.2 Devore

Ex 2: two coins, one biased one fair



$\mathbb{P}[X=1 \mid Y=0] = 0.5$, $\mathbb{P}[X=1 \mid Y=1] = 0.65$

$\boxed{\mathbb{E}[X \mid Y=y] = \sum_i x_i P_{X|Y}(x_i \mid y)}$ Conditional Mean 5.7 Leon

$\mathbb{E}[X \mid Y=0] = 1 \cdot \mathbb{P}[X=1 \mid Y=0] + 0 \cdot \mathbb{P}[X=0 \mid Y=0] = 0.5$

$\mathbb{E}[X \mid Y=1] = 1 \cdot \mathbb{P}[X=1 \mid Y=1] + 0 \cdot \mathbb{P}[X=0 \mid Y=1] = 0.65$

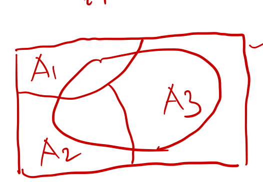
What about $\mathbb{E}[X]$?

$$\left.\begin{aligned} \mathbb{P}[X=1] &= \mathbb{P}[X=1, Y=0] + \mathbb{P}[X=1, Y=1] \\ &= \mathbb{P}[X=1 \mid Y=0]\,\mathbb{P}[Y=0] + \mathbb{P}[X=1 \mid Y=1]\,\mathbb{P}[Y=1] \end{aligned}\right\}$$

$= 0.5\,\mathbb{P}[Y=0] + 0.65\,\mathbb{P}[Y=1]$

The Law of Total Probability

$P(B) = \sum_{i=1}^{K} P(B \mid A_i)\, P(A_i)$



(2.4 Devore 9ed)

$\rightarrow$ $\mathbb{E}[X] = \mathbb{P}[X=1]$ is a function of the pmf of Y, i.e., $\{P_Y(y), y \in \mathbb{R}\}$, so $\mathbb{E}[X]$ is a value.

$\rightarrow$ In contrast, $\mathbb{E}[X \mid Y=y]$ is a fun of $\{P_{X|Y}(x \mid y), x \in \mathbb{R}\}$, or a fun of y.

Ex 3:

X \ Y	1	2	3
0	0.1	0.2	0.3
1	0.2	0.1	0.1

y	1	2	3
$p_Y(y)$	0.3	0.3	0.4

x	0	1
$p_X(x)$	0.6	0.4

Note $p_Y(y) \overset{def}{=} \mathbb{P}[Y=y]$

$\mathbb{E}[Y] = 1\times 0.3 + 2\times 0.3 + 3\times 0.4 = 0.3 + 0.6 + 1.2 = 2.1$

$\mathbb{E}[X] = 0\times 0.6 + 1\times 0.4 = 0.4$

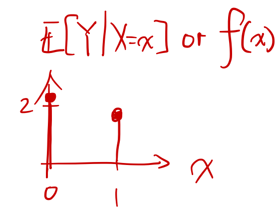
$p_{Y|X}(y|x)$

	y: 1	2	3
$x=0$	1/6	1/3	1/2
$x=1$	1/2	1/4	1/4

Note $p_{Y|X}(y|x) = \mathbb{P}[Y=y \mid X=x]$

$\mathbb{E}[Y|X=0] = 1\times\frac{1}{6} + 2\times\frac{1}{3} + 3\times\frac{1}{2} = \frac{1}{6}+\frac{4}{6}+\frac{9}{6} = \frac{7}{3}$

$\mathbb{E}[Y|X=1] = 1\times\frac{1}{2} + 2\times\frac{1}{4} + 3\times\frac{1}{4} = 1+\frac{3}{4} = \frac{7}{4}$



For the cts case:

Def: $\mathbb{E}[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$: value

$\mathbb{E}[Y|X=x] = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy$: a func of x

Note: $f_Y(y) = \int_{\mathbb{R}} f_{XY}(x,y)\, dx$

$f_{Y|X} = \frac{f_{XY}(x,y)}{f_X(x)}$

Ex 4 (Property of cond expectation):

$X \sim \text{Uniform}(0,10)$

$Y = \beta_0 + \beta_1 X + \beta_2 X^2 + e$, $e \sim N(0,1)$

$\mathbb{E}[X] = 5$ (value)

$\mathbb{E}[Y|X=x] = \mathbb{E}[\beta_0 + \beta_1 X + \beta_2 X^2 + e \mid X=x]$

Important property !!!

$= \mathbb{E}[\beta_0 + \beta_1 x + \beta_2 x^2 + e \mid X=x]$

$= \beta_0 + \beta_1 x + \beta_2 x^2 + \mathbb{E}[e \mid X=x]$

$= \beta_0 + \beta_1 x + \beta_2 x^2$ (A fun of x)

Proof: $\mathbb{E}[\overset{Y}{\widetilde{h(X)}} \mid X=x] = \int_{y\in\mathbb{R}} y\, f_{Y|X}(y|x)\, dy$

$= \int_y h(x)\, f_{Y|X}(y|x)\, dy$

$= h(x) \underbrace{\int_y f_{Y|X}(y|x)\, dy}_{1} = h(x)$

$\mathbb{E}[aX+b] = a\,\mathbb{E}X + b$ What's the intuition?

Btw, can you prove

$Var(aX+b) = a^2 Var(X)$ 3.3 Devore

$Var(X \pm Y) = Var(X) + Var(Y)$, $X \perp\!\!\!\perp Y$