ECE 792-41 Homework 1
Material Covered: Probability, Statistics, and Random Processes

Problem 1 (Conditional Probability and Law of Total Probability) There has been a great deal of controversy over the years regarding what types of surveillance are appropriate to prevent terrorism. Suppose that a particular surveillance system has a 99% chance of correctly identifying a future terrorist and a 99.9% chance of correctly identifying someone who is not a future terrorist. Suppose that there are $1,000$ future terrorists in a population of 300 million.

a) If one of these 300 million is randomly selected, scrutinized by the system, and identified as a future terrorist, what is the probability that he/she actually is a future terrorist?

b) Does the value of this probability make you surprise? If you are the designer of the surveillance system, which part(s) of the system needs to be improved in order to achieve a reasonable detection performance?

Problem 2 (Random Variable) Consider the pdf for total waiting time Y for two buses

$$f_Y(y) = \begin{cases} \frac{1}{25}y, & 0 \leq y < 5, \\ \frac{2}{5} - \frac{1}{25}y, & 5 \leq y \leq 10, \\ 0, & \text{otherwise.} \end{cases}$$

a) Compute and sketch the cdf of Y.

b) Obtain an expression for the $(100 \cdot p)$th percentile, $p \in [0, 1]$.

c) Compute $E(Y)$ and $V(Y)$. How do these compare with the expected waiting time and variance for a single bus when the time is uniformly distributed on $[0, 5]$?

Problem 3 (Random Variables and Random Processes) Let $X_i,\ i \in Z$ be pairwise uncorrelated random variables with mean μ and variance σ^2, i.e.,

$$\text{Cov}(X_i, X_j) = \sigma^2\, \delta[i-j] = \begin{cases} \sigma^2, & i = j, \\ 0, & i \neq j. \end{cases}$$

A linear filter computes the weighted average

$$Y_n = 3X_n - 2X_{n-1} + X_{n-2}. \tag{1}$$

a) Determine $E[Y_n]$.

b) Determine $\text{Cov}(X_n, Y_{n+k})$ for every integer k, and plot the result as a function of k.

c) Determine $V(Y_n)$ and $E[Y_n^2]$.

d) Determine $\mathrm{Cov}(Y_n, Y_{n+1})$, $\mathrm{Cov}(Y_n, Y_{n+2})$, and $\mathrm{Cov}(Y_n, Y_{n+k})$ for $k \geq 3$.

e) Plot $\mathrm{Cov}(Y_n, Y_{n+k})$ against k, where k ranges over all integers.

f) Now suppose that each filter output Y_n is corrupted by additive noise W_n:

$$Z_n = Y_n + W_n \tag{2}$$

where $\mathrm{Cov}(X_i, W_j) = 0$ for all (i, j) and $E[W_i] = 0$, $\mathrm{Cov}(W_i, W_j) = \tau^2\,\delta[i-j]$. Determine $\mathrm{Cov}(Z_n, Z_{n+k})$ for all (n, k).

Problem 4 (Unbiased Estimator) Prove that the sample variance $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is an unbiased estimator of σ^2, i.e., $E[s^2] = \sigma^2$.

Problem 5 (Method of Moments Estimator and Law of Large Numbers) You are given a random sample of size n drawn from a distribution of a random variable X with mean value μ and variance σ^2.

a) Find the method of moment estimators for μ and σ^2, denoted as $\hat{\mu}_n$ and $\widehat{\sigma_n^2}$, respectively.

b) What are the mean and variance of the estimator $\hat{\mu}_n$?

c) Explain why does the estimator $\hat{\mu}_n$ become more accurate as the sample size n increases.

d) Prove that the estimator $\hat{\mu}_n$ is consistent.

Problem 6 (Maximum Likelihood Estimator (MLE)) Calculate the MLE of θ from a sample of size n drawn from the normal distribution as follows:

$$f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}}\, e^{-x^2/2\theta}, \quad -\infty < x < \infty. \tag{3}$$

Problem 7 (MLE for a Customized Experiment) Each of n specimens is to be weighed twice on the same scale. Let X_i and Y_i denote the two observed weights for the ith specimen. Suppose X_i and Y_i are independent of one another, each normally distributed with mean value μ_i (the true weight of specimen i) and variance σ^2.

a) Show that the MLE of σ^2 is $\hat{\sigma}^2 = \sum_{i=1}^{n}(X_i - Y_i)^2/(4n)$. [Hint: You need to set simultaneously $\partial L/\partial \mu_k = 0,\ k = 1, \cdots, n$ and $\partial L/\partial \sigma^2 = 0$.]

b) Is the MLE $\hat{\sigma}^2$ an unbiased estimator of σ^2? Find an unbiased estimator of σ^2. [Hint: For any rv Z, $E(Z^2) = V(Z) + [E(Z)]^2$. Apply this to $Z = X_i - Y_i$.]

Problem 8 (Invariance Principle of MLE) The shear strength of each of ten test spot welds is determined, yielding the following data (psi): 392 376 401 367 389 362 409 415 358 375.

a) Assuming that shear strength is normally distributed, estimate the true average shear strength and standard deviation of shear strength using the method of maximum likelihood. [You may use theoretical results from the lecture and/or from other problems of this homework.]

b) Again assuming a normal distribution, estimate the strength value below which 95% of all welds will have their strengths. [Hint: What is the 95th percentile in terms of μ and σ? Now use the invariance principle.]

c) Suppose we decide to examine another test spot weld. Let X = shear strength of the weld. Use the given data to obtain the MLE of $P(X \leq 400)$. [Hint: $P(X \leq 400) = \Phi\left((400 - \mu)/\sigma\right)$.]