ECE 792-41 Homework 4

Material Covered: Spectrum Estimation, ACF Estimator, Durbin's Method, Order Selection

Problem 1 In this problem, we show that the periodogram is an inconsistent estimator by examining the estimator at $f = 0$:

$$\hat{P}_{\text{PER}}(0) = \frac{1}{N}\left(\sum_{n=0}^{N-1} x[n]\right)^2.$$

If $x[n]$ is a real white Gaussian noise process with PSD

$$P_{xx}(f) = \sigma_x^2$$

find the mean and variance of $\hat{P}_{\text{PER}}(0)$. Does the variance converge to zero as $N \to \infty$? *Hint:* Note that

$$\hat{P}_{\text{PER}}(0) = \sigma_x^2 \left(\sum_{n=0}^{N-1} \frac{x[n]}{\sigma_x\sqrt{N}}\right)^2$$

where the quantity inside the parentheses is $\sim N(0,1)$.

Problem 2 Consider the estimator

$$\hat{P}_{\text{AV-PER}}(0) = \frac{1}{N}\sum_{m=0}^{N-1} \hat{P}^{(m)}_{\text{PER}}(0)$$

where

$$\hat{P}^{(m)}_{\text{PER}}(0) = x^2[m]$$

for the process from Problem 1. This estimator may be viewed as an *averaged periodogram.* In essence the data record is sectioned into blocks (in this case, of length 1) and the periodograms for each block are averaged. Find the mean and variance of $\hat{P}_{\text{AV-PER}}(0)$. Compare this result to that obtained in Problem 1.

Problem 3 Find the variance of the unbiased ACF estimator

$$\hat{r}'_{xx}[k] = \tfrac{1}{N-k}\textstyle\sum_{n=0}^{N-1-k} x[n]x[n+k] \quad 0 \le k \le N-1$$

for real data which is a zero-mean white Gaussian process with variance σ_x^2. What happens as the lag k increases? Using the variance of the unbiased ACF estimator you obtained, find the variance of the biased ACF estimator

$$\hat{r}_{xx}[k] = \frac{1}{N} \sum_{n=0}^{N-1-k} x[n]x[n+k]$$

without going through the derivation again. What happens as the lag k increases?

Hint: With the help of the Isserlis' theorem, first prove that for any real zero-mean Gaussian process the variance of the unbiased ACF estimator is

$$V(\hat{r}'_{xx}[k]) = \frac{1}{N-k} \sum_{j=-(N-1-k)}^{N-1-k} \left(1 - \frac{|\, j \,|}{N-k}\right) (r_{xx}^2[j] + r_{xx}[j+k]r_{xx}[j-k]).$$

Problem 4 Implement the Durbin's method for estimating an order-4 moving average model. Let the maximum allowed order of the approximated model L to be $8, 16, ...,$ or 1024. For each L, repeat the calculation for the estimated psd and estimated MA coefficients 1000 times, and store the results for the tasks below. You may reuse the Levinson-Durbin recursion code from your project.

(a) Draw in one plot a family of 8 MSE curves for estimated psd against frequency. Each MSE curve corresponds to a specific value of L, and should be the averaged MSE of 1000 realizations. Properly label the curves with Matlab function `legend`.

(b) Draw in one plot a family of 8 MSE curves of estimated MA coefficients against the index of the coefficient. Properly label the curves.

(c) For each MA coefficient, draw one error-bar plot against L using Matlab function `errorbar`. Sample mean and sample standard deviation should be supplied as parameters `y` and `err` of the `errorbar`. L-axis should be in the log10 scale.

(d) Summarize the effect of L based on the plots in (a)–(c).

Problem 5 Generate a length-10000 AR(2) signal. What are the selected orders by AIC, MDL, 10-fold cross-validation, and leave-one-out cross-validation? Note that you should use Yule-Walker equations for estimating the AR coefficients. For the purpose of cross-validation, you may consider cutting the AR signal into 50 non-overlapped segments of length 200, and consider each segment as a data point. Explain how you calculate $\hat{r}(k)$ during the cross-validation. Comment on the results you obtained. Repeat the problem for a length-10000 AR(10) signal. You may reuse the AR process related code from your project.