Parametric Signal Modeling and Linear Prediction Theory 4. The Levinson-Durbin Recursion

Electrical & Computer Engineering
North Carolina State University

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Contact: *chauwai.wong@ncsu.edu*. Updated: February 14, 2019.

Complexity in Solving Linear Prediction

(Refs: Hayes §5.2; Haykin 4th Ed. §3.3)

Recall Augmented Normal Equation for linear prediction:

$$\underline{\text{FLP}} \quad \mathbf{R}_{M+1}\underline{a}_M = \begin{bmatrix} P_M \\ \underline{0} \end{bmatrix} \qquad \underline{\text{BLP}} \quad \mathbf{R}_{M+1}\underline{a}_M^{B^*} = \begin{bmatrix} \underline{0} \\ P_M \end{bmatrix}$$

As $\mathbf{R}_{M+1}$ is usually non-singular, $\underline{a}_M$ may be obtained by inverting $\mathbf{R}_{M+1}$, or Gaussian elimination for solving equation array:

$\Rightarrow$ Computational complexity $O(M^3)$.

Exploiting Structures in Matrix and LP Problem

Complexity in solving a general linear equation array:

- Method-1: invert the matrix, e.g. compute determinant of $\mathbf{R}_{M+1}$ matrix and the adjacency matrices

  $\Rightarrow$ matrix inversion has $O(M^3)$ complexity

- Method-2: use Gaussian elimination

  $\Rightarrow$ approximately $M^3/3$ multiplication and division

By exploring the Toeplitz structure of the matrix, Levinson-Durbin recursion can reduce complexity to $O(M^2)$

- M steps of order recursion, each step has a linear complexity w.r.t. intermediate order

- Memory use: Gaussian elimination $O(M^2)$ for the matrix, vs. Levinson-Durbin $O(M)$ for the autocorrelation vector and model parameter vector.

Levinson-Durbin Recursion

The **Levinson-Durbin recursion** is an order-recursion to efficiently solve linear systems with Toeplitz matrices, e.g., Augmented N.E.

> M steps of order recursion, each step has a linear complexity w.r.t. intermediate order

The recursion can be stated in two ways for the linear prediction problem:

1. Forward prediction point of view
2. Backward prediction point of view

Two Points of View of LD Recursion

Denote $\underline{a}_m \in \mathbb{C}^{(m+1)\times 1}$ as the tap weight vector of a forward-prediction-error filter of order $m = 0, ..., M$.

$a_{m-1,0} = 1$, $a_{m-1,m} \triangleq 0$, $a_{m,m} = \Gamma_m$ (a constant "**reflection coefficient**")

Forward prediction point of view

$$a_{m,k} = a_{m-1,k} + \Gamma_m \, a^*_{m-1,m-k}, \; k = 0, 1, \ldots, m$$

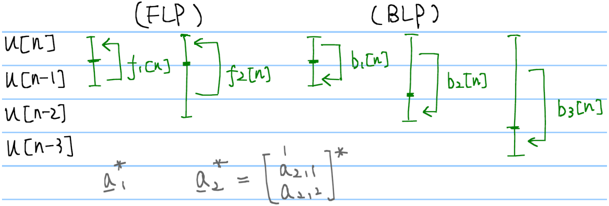
In vector form: $\underline{a}_m = \begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix} + \Gamma_m \begin{bmatrix} 0 \\ \underline{a}^{B*}_{m-1} \end{bmatrix}$ $(**)$

Backward prediction point of view

$$a^*_{m,m-k} = a^*_{m-1,m-k} + \Gamma^*_m \, a_{m-1,k}, \; k = 0, 1, \ldots, m$$

In vector form: $\underline{a}^{B*}_m = \begin{bmatrix} 0 \\ \underline{a}^{B*}_{m-1} \end{bmatrix} + \Gamma^*_m \begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix}$ (can be obtained by reordering and conjugating $(**)$)

Recall: Forward and Backward Prediction Errors



- $f_m[n] = u[n] - \hat{u}[n] = \underline{a}_m^H \underbrace{\underline{u}[n]}_{(m+1)\times 1}$
- $b_m[n] = u[n-m] - \hat{u}[n-m] = \underline{a}_m^{B,T}\, \underline{u}[n]$

(3) Verify the Update Equations of the LD Recursion

Left multiply both sides of $(**)$ by $\mathbf{R}_{m+1}$:

LHS: $\mathbf{R}_{m+1}\underline{a}_m = \begin{bmatrix} P_m \\ \underline{0}_m \end{bmatrix}$ (by augmented N.E.)

RHS (1): $\mathbf{R}_{m+1}\begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_m & \underline{r}_m^{B^*} \\ \underline{r}_m^{BT} & r(0) \end{bmatrix}\begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} \mathbf{R}_m\underline{a}_{m-1} \\ \underline{r}_m^{BT}\underline{a}_{m-1} \end{bmatrix} = \begin{bmatrix} P_{m-1} \\ \underline{0}_{m-1} \\ \Delta_{m-1} \end{bmatrix} \text{ where } \Delta_{m-1} \triangleq \underline{r}_m^{BT}\underline{a}_{m-1}$$

RHS (2): $\mathbf{R}_{m+1}\begin{bmatrix} 0 \\ \underline{a}_{m-1}^{B*} \end{bmatrix} = \begin{bmatrix} r(0) & \underline{r}^H \\ \underline{r} & \mathbf{R}_m \end{bmatrix}\begin{bmatrix} 0 \\ \underline{a}_{m-1}^{B*} \end{bmatrix}$

$$= \begin{bmatrix} \underline{r}^H\underline{a}_{m-1}^{B*} \\ \mathbf{R}_m\underline{a}_{m-1}^{B*} \end{bmatrix} = \begin{bmatrix} \Delta_{m-1}^* \\ \underline{0}_{m-1} \\ P_{m-1} \end{bmatrix}$$

Computing Γ_m

Put together LHS and RHS: for the order update recursion $(**)$ to hold, we should have

$$\begin{bmatrix} P_m \\ \underline{0}_m \end{bmatrix} = \begin{bmatrix} P_{m-1} \\ \underline{0}_{m-1} \\ \Delta_{m-1} \end{bmatrix} + \Gamma_m \begin{bmatrix} \Delta^*_{m-1} \\ \underline{0}_{m-1} \\ P_{m-1} \end{bmatrix}$$

$$\Rightarrow \begin{cases} P_m = P_{m-1} + \Gamma_m \Delta^*_{m-1} \\ 0 = \Delta_{m-1} + \Gamma_m P_{m-1} \end{cases}$$

$\Rightarrow$

$$a_{m,m} = \Gamma_m = -\frac{\Delta_{m-1}}{P_{m-1}}$$

$$P_m = P_{m-1}\left(1 - |\Gamma_m|^2\right)$$

Caution: Do not confuse the power term P_m and the ratio term Γ_m.

(4) Reflection Coefficients Γ_m

To ensure the prediction MSE $P_m \geq 0$ and P_m non-increasing as we increase the order of the predictor (i.e., $0 \leq P_m \leq P_{m-1}$), we require $|\Gamma_m|^2 \leq 1$, $\forall m > 0$.

Let $P_0 = r(0)$ as the initial estimation error has power equal to the signal power (i.e., no regression is applied), we have

$$P_M = P_0 \cdot \prod_{m=1}^{M} (1 - |\Gamma_m|^2)$$

Question: Under what situation $\Gamma_m = 0$?
i.e., increasing order won't reduce error.

Consider a process with Markovian-like property in 2nd order statistic sense (e.g. AR process) s.t. info of further past is contained in k recent samples

(5) About Δ_m

One can show that the cross-correlation of <u>BLP error</u> and <u>FLP error</u> $\mathbb{E}\left[b_{m-1}[n-1]f^*_{m-1}[n]\right]$ is equal to Δ_{m-1}.

(Derive from the definition $\Delta_{m-1} \triangleq \underline{r}_m^{BT}\underline{a}_{m-1}$, and use definitions of $b_{m-1}[n-1], f^*_{m-1}[n]$ and orthogonality principle.)

Thus the reflection coefficient can be written as

$$\Gamma_m = -\frac{\Delta_{m-1}}{P_{m-1}} = -\frac{\mathbb{E}\left[b_{m-1}[n-1]f^*_{m-1}[n]\right]}{\mathbb{E}\left[|f_{m-1}[n]|^2\right]}$$

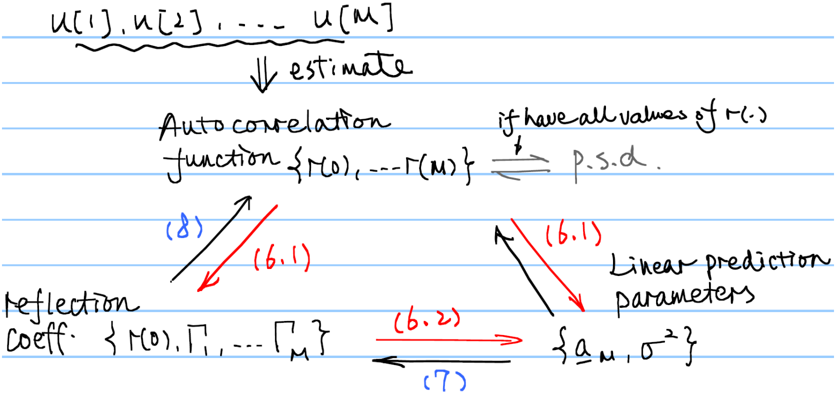
which is also the negative *partial correlation coefficient*.

Note: for the 0th order predictor, use the mean value, i.e., zero, as the estimate, s.t. $f_0[n] = u[n] = b_0[n]$,

$\therefore \Delta_0 = \mathbb{E}\left[b_0[n-1]f_0^*[n]\right] = \mathbb{E}\left[u[n-1]u^*[n]\right] = r(-1) = r^*(1)$

Preview: Relations of w.s.s and LP Parameters

For any w.s.s. process $\{u[n]\}$:



(6) Computing $\underline{a}_M$ and P_M by Forward Recursion

Case-1 : If we know the autocorrelation function $r(\cdot)$:

① $\Delta_0 = r(-1)$, $P_0 = r(0)$

② for $m = 1, \ldots, M$ (order recursion)

$\Gamma_m = -\dfrac{\Delta_{m-1}}{P_{m-1}}$

for $k = 1, \ldots, m$ (diff predictor parameters for order-m)

$a_{m,k} = a_{m-1,k} + \Gamma_m a^*_{m-1,m-k}$

(where $a_{m-1,0} = 1$; $a_{m-1,m} = 0$)

$\Delta_m = \underline{r}_{m+1}^{BT} \underline{a}_m$

$P_m = P_{m-1}(1 - |\Gamma_m|^2)$

- # of iterations $= \sum_{m=1}^{M} m = \frac{M(M+1)}{2}$, comp. complexity is $O(M^2)$
- $r(k)$ may be estimated from time average of one realization of $\{u[n]\}$:

  $\hat{r}(k) = \frac{1}{N-k} \sum_{n=k+1}^{N} u[n]u^*[n-k], \; k = 0, 1, \ldots, M$
  (recall correlation ergodicity)

(6) Computing $\underline{a}_M$ and P_M by Forward Recursion

Case-2 : If we know Γ_1, Γ_2, ..., Γ_M and $P_0 = r(0)$, we can carry out the recursion for $m = 1, 2, \ldots, M$:

$$\begin{cases} a_{m,k} = a_{m-1,k} + \Gamma_m a^*_{m-1,m-k}, \; k = 1, \ldots, m \\ P_m = P_{m-1}\left(1 - |\Gamma_m|^2\right) \end{cases}$$

Note: $a_{m,m} = a_{m-1,m} + \Gamma_m a^*_{m-1,0} = 0 + \Gamma_m \cdot 1 = \Gamma_m$

(7) Inverse Form of Levinson-Durbin Recursion

Given the tap-weights $\underline{a}_M$, find the reflection coefficients $\Gamma_1, \Gamma_2, \ldots, \Gamma_M$:

$$\text{Recall: } \begin{cases} \text{(FP) } a_{m,k} = a_{m-1,k} + \Gamma_m \, a^*_{m-1,m-k}, \ \ k = 0, \ldots, m \\ \text{(BP) } a^*_{m,m-k} = a^*_{m-1,m-k} + \Gamma^*_m \, a_{m-1,k}, \ \ a_{m,m} = \Gamma_m \end{cases}$$

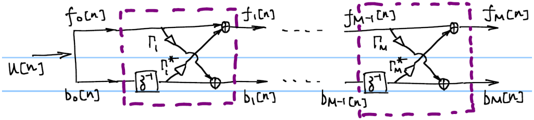
Multiply (BP) by Γ_m and subtract from (FP):

$$a_{m-1,k} = \frac{a_{m,k} - \Gamma_m a^*_{m,m-k}}{1 - |\Gamma_m|^2} = \frac{a_{m,k} - a_{m,m} a^*_{m,m-k}}{1 - |a_{m,m}|^2}, k = 0, \ldots, m-1$$

$\Rightarrow \Gamma_m = a_{m,m}$, $\Gamma_{m-1} = a_{m-1,m-1}, \ldots,$ i.e., From $\underline{a}_M \Rightarrow \underline{a}_m \Rightarrow \Gamma_m$

iterate with $m = M-1, M-2, \ldots$ *to lower order*

see §5 Lattice structure:



(8) Autocorrelation Function & Reflection Coefficients

Recall: The 2nd-order statistics of a stationary time series can be represented in terms of autocorrelation function $r(k)$, or equivalently the power spectral density by taking DTFT.

Another way is to use $\{r(0), \Gamma_1, \Gamma_2, \ldots, \Gamma_M\}$.

To find the relation between them, recall:

$\Delta_{m-1} \triangleq \underline{r}_m^{BT} \underline{a}_{m-1} = \sum_{k=0}^{M-1} a_{m-1,k} r(-m+k)$ and $\Gamma_m = -\frac{\Delta_{m-1}}{P_{m-1}}$

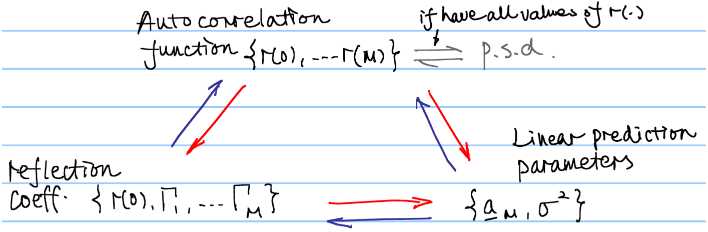
$\Rightarrow -\Gamma_m P_{m-1} = \sum_{k=0}^{m-1} a_{m-1,k} r(k-m)$, where $a_{m-1,0} = 1$.

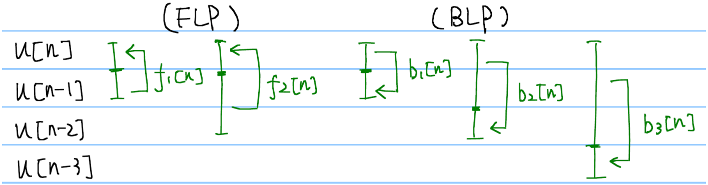
(8) Autocorrelation Function & Reflection Coefficients

1. $r(m) = r^*(-m) = -\Gamma_m^* P_{m-1} - \sum_{k=1}^{m-1} a_{m-1,k}^* r(m-k)$
   Given $r(0), \Gamma_1, \Gamma_2, \ldots, \Gamma_M$, can get $\underline{a}_m$ using Levinson-Durbin recursion s.t. $r(1), \ldots, r(M)$ can be generated recursively.
2. Recall if $r(0), \ldots, r(M)$ are given, we can get $\underline{a}_m$.
   So $\Gamma_1, \ldots, \Gamma_M$ can be obtained recursively: $\Gamma_m = a_{m,m}$
3. These facts imply that the reflection coefficients $\{\Gamma_k\}$ can uniquely represent the 2nd-order statistics of a w.s.s. process.

Summary

Statistical representation of w.s.s. process





Detailed Derivations/Examples

Example of Forward Recursion Case-2

e.g. (case 2). Given $\Gamma_1, \Gamma_2, \Gamma_3$ and $P(0)$, find $\underline{a}_3$ and P_3 of a prediction-error filter of order 3.

⓪ $P_0 = r(0)$

① $m=1$: $a_{1,0}=1$; $a_{1,1}=\Gamma_1$; $a_{1,2}=0$; $P_1 = P_0(1-|\Gamma_1|^2)$

② $m=2$: $a_{2,0}=1$; $a_{2,1}=a_{1,1}+\Gamma_2 a_{1,1}^* = \Gamma_1 + \Gamma_2 \cdot \Gamma_1^*$ (used in §2.5.4. for inverse filtering)

$a_{2,2}=\Gamma_2$

$P_2 = P_1(1-|\Gamma_2|^2)$

③ $m=3$: $a_{3,0}=1$; $a_{3,1}=a_{2,1}+\Gamma_3 a_{2,2}^* = \Gamma_1+\Gamma_2\Gamma_1^* + \Gamma_3 \cdot \Gamma_2^*$

$a_{3,2} = a_{2,2}+\Gamma_3 a_{2,1}^* = \Gamma_2 + \Gamma_3\Gamma_1^* + \Gamma_1\Gamma_2^*\Gamma_3$

$a_{3,3} = \Gamma_3$

$P_3 = P_2(1-|\Gamma_3|^2)$

Proof for Δ_{m-1} Property

Proof: In HW#4

$$\Delta_{m-1} = \Gamma_m^{BT} a_{m-1} = [r(-m), \dots r(-1)]\, a_{m-1}$$
$$= E[u^*[n]\, u_m^{BT}[n-1]]\, a_{m-1}$$
$$= E[u^*[n]\, u_m^{BT}[n-1]\, a_{m-1}]$$
$$= E[u^*[n]\, (u_m^T[n-1]\, a_{m-1}^B)]$$
$$= E[u^*[n]\, b_{m-1}[n-1]]$$
$$= E[f_{m-1}^*[n]\, b_{m-1}[n-1]]$$

Recall

① $\Gamma_M = \begin{bmatrix} r(-1) \\ \vdots \\ r(-M) \end{bmatrix}$

② $r(-k) = E[u[n-k]\, u^*[n]]$
$= (E[u[n]\, u^*[n-k]])^*$

③ $u_m[n-1] = \begin{bmatrix} u[n-1] \\ \vdots \\ u[n-m] \end{bmatrix}$

④ $b_{m-1}[n-1] = \sum_{k=0}^{m-1} a_{m-1,m-1-k}\, u[n-1-k]$
$= a_{m-1}^B\, u_m[n-1]$

⑤ $a_{m-1,0} = 1$
$f_{m-1}[n] = \sum_{k=0}^{m-1} a_{m-1,k}^*\, u[n-k]$
$= u[n] + \sum_{k=1}^{m-1} a_{m-1,k}^*\, u[n-k]$

⑥ $b_{m-1}[n-1] \perp \{u[n-1], \dots u[n-m+1]\}$

Haykin's 4th Ed. (P152)

* Partial correlation (PARCOR) coeff. between $f_{m-1}[n]$ and $b_{m-1}[n-1]$.

$$\rho_m \triangleq \frac{E[b_{m-1}[n-1] f_{m-1}^*[n]]}{(E[|b_{m-1}[n-1]|^2]\, E[|f_{m-1}[n]|^2])^{1/2}} \overset{\text{for w.s.s.}}{=} \frac{\Delta_{m-1}}{P_{m-1}} = -\Gamma_m$$

Recall $E[|f_m[n]|^2] = E[|b_m[n]|^2] = P_m$