ECE 792-41 Homework 5
Material Covered: ARMA Processes, Yule–Walker Equations, Autocorrelation Functions, Wiener Filter.

Problem 1 A first-order real-valued autoregressive (AR) process $\{u(n)\}$ satisfies the following difference equation

$$u(n) + a_1 u(n-1) = v(n)$$

where a_1 is a constant and $\{v(n)\}$ is a white-noise process with mean-value function m_v and variance σ_v^2. Such a process is also referred to as a *first-order Markov process*.

(a) Suppose in practical implementation, the generation of the process $\{u(n)\}$ starts at $n = 1$ with initialization $u(0) = 0$. Determine the mean-value function of the simulated process $\{u'(n)\}$. Under what conditions does it diverge? Under what conditions does it converge to a constant and what the constant is?

(b) Now consider $m_v = 0$. Determine the variance of the actual process $\{u'(n)\}$. Under what conditions does $\mathrm{Var}[u'(n)]$ converge to a constant and what the constant is?

(c) For the conditions specified in part (b), find the autocorrelation function of the process $\{u'(n)\}$. Sketch this autocorrelation function when $n \gg k$, for the two cases $0 < a_1 < 1$ and $-1 < a_1 < 0$. (Hint: You may want to proceed with $k > 0$ and $k \le 0$ separately.)

Problem 2

(a) Consider an MA process $\{x(n)\}$ of order 2 described by the difference equation

$$x(n) = v(n) + 0.75\, v(n-1) + 0.25\, v(n-2)$$

where $\{v(n)\}$ is a zero mean white noise process of unit variance. The requirement is to approximate the process by an AR process $\{u(n)\}$ of order M. Do this approximation for the orders $M = 2$ and $M = 5$, respectively.

(b) Consider an autoregressive process $\{u(n)\}$ of order 2, described by the difference equation

$$u(n) = u(n-1) - 0.5\, u(n-2) + v(n)$$

where $\{v(n)\}$ is a white-noise process of zero mean and variance 0.5.

(i) Write two Yule–Walker equations for the process. (No derivation is needed.)

(ii) Solve these two equations for the autocorrelation function values $r(1)$ and $r(2)$.

(iii) Find the variance of $\{u(n)\}$.

Problem 3 Let a real-valued AR(2) process $\{x(n)\}$ be described by

$$u(n) = x(n) + a_1 x(n-1) + a_2 x(n-2)$$

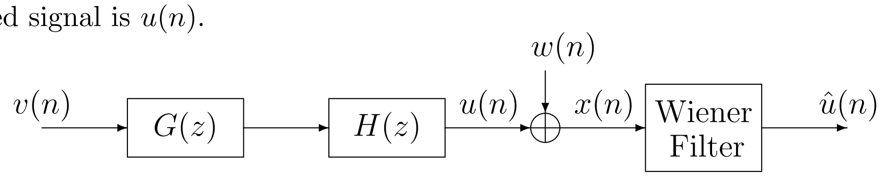
where $u(n)$ is a white noise of zero mean and variance σ^2, and $u(n)$ is uncorrelated with past values $x(n-1)$, $x(n-2)$.

(a) Evaluate $r(k)$ starting from $\mathbb{E}\left[x(n+k)x^*(n)\right]$, for $k = 0, 1, 2$. The results should be in terms of $r(\cdot)$ and σ^2. Compare these results with Yule-Walker Equations. What can you conclude?

(b) Write $r(1)$ and $r(2)$ in terms of $r(0)$.

(c) What's the analytic form of the variance of the process $\{x(n)\}$?

Problem 4 Assume $v(n)$ and $w(n)$ are white Gaussian random processes with zero mean and variance 1. The two filters shown in the figure below are $G(z) = \frac{1}{1-0.4z^{-1}}$ and $H(z) = \frac{2}{1-0.5z^{-1}}$. The desired signal is $u(n)$.



(a) Design a 1st-order Wiener filter. Derive the analytic form of the prediction error, $J = \mathbb{E}\left[|u(n) - \hat{u}(n)|^2\right]$, and then evaluate.

(b) Repeat (a) by using a 2nd-order Wiener filter.

(c) Argue why the results in (a) and (b) are different.

Problem 5 The tap-input vector of a transversal filter is defined by

$$\mathbf{u}(n) = \alpha(n)\mathbf{s}(\omega) + \mathbf{v}(n)$$

where

$$\begin{aligned} \mathbf{s}(\omega) &= [1, e^{-j\omega}, \ldots, e^{-j\omega(M-1)}]^T \\ \mathbf{v}(n) &= [v(n), v(n-1), \ldots, v(n-M+1)]^T \end{aligned}$$

i.e., $u(n-k) = \alpha(n) \cdot e^{-jk\omega} + v(n-k)$ for $k = 0, \ldots, M-1$. For the tap-input vector at a given time n, $\alpha(n)$ is a complex random variable with zero mean and variance $\sigma_\alpha^2 = \mathbb{E}[|\alpha(n)|^2]$, and $\alpha(n)$ is uncorrelated with the w.s.s. process $\mathbf{v}(n)$.

(a) Determine the correlation matrix of the tap-input vector $\mathbf{u}$(n).

(b) Suppose that the desired response $d(n)$ is uncorrelated with $\mathbf{u}$(n). What is the value of the tap-weight vector of the corresponding Wiener filter?

(c) Suppose that the variance σ_α^2 is zero, and the desired response is defined by

$$d(n) = v(n-k)$$

where $0 \leq k \leq M-1$. What is the new value of the tap-weight vector of the Wiener filter?

(d) Determine the tap-weight vector of the Wiener filter for a desired response defined by

$$d(n) = \alpha(n)e^{-j\omega\tau}$$

where τ is a prescribed delay.

Problem 6 We explore an application of Wiener filtering to radar. The sampled form of the transmitted radar signal is $A_0 e^{j\omega_0 n}$ where ω_0 is the transmitted angular frequency, and A_0 is the transmitted complex amplitude. The received signal is

$$u(n) = A_1 e^{j\omega_1 n} + v(n)$$

where $|A_1| < |A_0|$ and ω_1 differs from ω_0 by virtue of the Doppler shift produced by the motion of a target of interest, and white noise $\{v(n)\}$ is uncorrelated with A_1.

(a) Show that the correlation matrix of $\{u(n)\}$, made up of M elements, may be written as

$$\mathbf{R} = \sigma_v^2 \mathbf{I} + \sigma_1^2 \mathbf{s}(w_1)\mathbf{s}^H(w_1)$$

where σ_v^2 is the variance of the zero-mean white noise $v(n)$, and

$$\begin{aligned} \sigma_1^2 &= \mathbb{E}[|A_1|^2] \\ \mathbf{s}(\omega_1) &= [1, e^{-j\omega_1}, \ldots, e^{-j\omega_1(M-1)}]^T. \end{aligned}$$

What is $\mathbf{R}^{-1}$?

(b) The time series $\{u(n)\}$ is applied to an M-tap Wiener filter with the cross-correlation vector $\mathbf{p}$ between $\{u(n)\}$ and the desired response $d(n)$ preset to

$$\mathbf{p} = \sigma_0^2\, \mathbf{s}(\omega_0)$$

where

$$\begin{aligned} \sigma_0^2 &= E[|A_0|^2] \\ \mathbf{s}(\omega_0) &= [1, e^{-j\omega_0}, \ldots, e^{-j\omega_0(M-1)}]^T \end{aligned}$$

Derive an expression for the tap-weight vector of the Wiener filter.

Hint: You may want to use the matrix inversion lemma:

$$(B^{-1} + CD^{-1}C^H)^{-1} = B - BC(D + C^H BC)^{-1} C^H B.$$