ECE 792-41 Homework 6

Material Covered: Lagrange Multipliers, Spectrum Estimation, ACF Estimator, Durbin's Method, Order Selection

Problem 1* A linear array consists of M uniformly spaced sensors. The individual sensor outputs are weighted and then summed, producing the output

$$e(n) = \sum_{k=1}^{M} \omega_k^* u_k(n)$$

where $u_k(n)$ is the output of sensor k at time n, and ω_k is the associated weight. The weights are chosen to minimize the mean-square value of $e(n)$, subject to the constraint

$$\mathbf{w}^H \mathbf{s} = 1$$

where $\mathbf{s}$ is a prescribed steering vector. By using the method of Lagrange multipliers, show that the optimum value of the vector $\mathbf{w}$ is

$$\mathbf{w}_0 = \frac{\mathbf{R}^{-1}\mathbf{s}}{\mathbf{s}^H \mathbf{R}^{-1}\mathbf{s}}$$

where $\mathbf{R}$ is the spatial correlation matrix of the linear array.

Hint: Construct a Lagrange function that is real valued. Let $f(\mathbf{w})$ be the expression of the constraint. You may construct a real valued version of the constraint expression by $\text{Re}\{2f(\mathbf{w})\} = f(\mathbf{w}) + f^*(\mathbf{w})$. Recall we discussed in lecture that when taking partial derivative, consider $\mathbf{w}$ and $\mathbf{w}^*$ as independent parameters.

Problem 2* Consider a discrete-time stochastic process $\{u(n)\}$ that consists of K (uncorrelated) complex sinusoids plus additive white noise of zero mean and variance σ^2. That is,

$$u(n) = \sum_{k=1}^{K} A_k e^{j\omega_k n} + v(n)$$

where the terms $A_k e^{j\omega_k n}$ and $v(n)$ refer to the kth sinusoid and noise, respectively. The process $\{u(n)\}$ is applied to a transversal filter with M taps, producing the output

$$e(n) = \mathbf{w}^H \mathbf{u}(n)$$

Assume that $M > K$. The requirement is to choose the tap-weight vector $\mathbf{w}$ so as to minimize the mean-square value of $e(n)$, subject to the multiple signal-protection constraint

$$\mathbf{S}^H \mathbf{w} = \mathbf{D}^{\frac{1}{2}} \mathbf{1}$$

where $\mathbf{S}$ is the $M \times K$ signal matrix whose k^{th} column has 1, $e^{j\omega_k}, \ldots, e^{j\omega_k(M-1)}$ for its elements, $\mathbf{D}$ is the $K \times K$ diagonal matrix whose nonzero elements equal the average powers of the individual sinusoids, and the $K \times 1$ vector $\mathbf{1}$ has 1's for all its elements. Using the method of Lagrange multipliers, show that the value of the optimum weight vector that results from the constraint optimization equals

$$\mathbf{w}_0 = \mathbf{R}^{-1}\mathbf{S}(\mathbf{S}^H \mathbf{R}^{-1} \mathbf{S})^{-1} \mathbf{D}^{\frac{1}{2}} \mathbf{1}$$

where $\mathbf{R}$ is the correlation matrix of the $M \times 1$ tap-input vector $\mathbf{u}(n)$.

[Note: The vector $\mathbf{w_0}$ in the previous problem is known as the spatial version of the Minimum Variance Distortionless Response (MVDR) from the array signal processing literature. And the result of $\mathbf{w_0}$ in this problem represents a temporal generalization of the MVDR formula. For more details on MVDR beamforming method, refer to Haykin's book *Adaptive Filter Theory.*]

Problem 3* In this problem, we show that the periodogram is an inconsistent estimator by examining the estimator at $f = 0$:

$$\hat{P}_{\text{PER}}(0) = \frac{1}{N}\left(\sum_{n=0}^{N-1} x[n]\right)^2.$$

If $x[n]$ is a real white Gaussian noise process with PSD

$$P_{xx}(f) = \sigma_x^2$$

find the mean and variance of $\hat{P}_{\text{PER}}(0)$. Does the variance converge to zero as $N \to \infty$? *Hint:* Note that

$$\hat{P}_{\text{PER}}(0) = \sigma_x^2 \left(\sum_{n=0}^{N-1} \frac{x[n]}{\sigma_x \sqrt{N}}\right)^2$$

where the quantity inside the parentheses is $\sim N(0,1)$.

Problem 4* Consider the estimator

$$\hat{P}_{\text{AV-PER}}(0) = \frac{1}{N} \sum_{m=0}^{N-1} \hat{P}_{\text{PER}}^{(m)}(0)$$

where

$$\hat{P}_{\text{PER}}^{(m)}(0) = x^2[m]$$

for the process from the previous problem. This estimator may be viewed as an *averaged periodogram*. In essence the data record is sectioned into blocks (in this case, of length 1) and the periodograms for each block are averaged. Find the mean and variance of $\hat{P}_{\text{AV-PER}}(0)$. Compare this result to that obtained in the previous problem.

Problem 5* Find the variance of the unbiased ACF estimator

$$\hat{r}'_{xx}[k] = \tfrac{1}{N-k} \textstyle\sum_{n=0}^{N-1-k} x[n]x[n+k] \quad 0 \le k \le N-1$$

for real data which is a zero-mean white Gaussian process with variance σ_x^2. What happens as the lag k increases? Using the variance of the unbiased ACF estimator you obtained, find the variance of the biased ACF estimator

$$\hat{r}_{xx}[k] = \frac{1}{N} \sum_{n=0}^{N-1-k} x[n]x[n+k]$$

without going through the derivation again. What happens as the lag k increases?

Hint: With the help of the Isserlis' theorem, first prove that for any real zero-mean Gaussian process the variance of the unbiased ACF estimator is

$$V(\hat{r}'_{xx}[k]) = \frac{1}{N-k} \sum_{j=-(N-1-k)}^{N-1-k} \left(1 - \frac{|\, j \,|}{N-k}\right) (r_{xx}^2[j] + r_{xx}[j+k]r_{xx}[j-k]).$$

Problem 6* Implement the Durbin's method for estimating an order-4 moving average model. Let the maximum allowed order of the approximated model L to be $8, 16, ...,$ or 1024. For each L, repeat the calculation for the estimated psd and estimated MA coefficients 1000 times, and store the results for the tasks below. You may reuse the Levinson-Durbin recursion code from your project.

(a) Draw in one plot a family of 8 MSE curves for estimated psd against frequency. Each MSE curve corresponds to a specific value of L, and should be the averaged MSE of 1000 realizations. Properly label the curves with Matlab function `legend`.

(b) Draw in one plot a family of 8 MSE curves of estimated MA coefficients against the index of the coefficient. Properly label the curves.

(c) For each MA coefficient, draw one error-bar plot against L using Matlab function `errorbar`. Sample mean and sample standard deviation should be supplied as parameters `y` and `err` of the `errorbar`. L-axis should be in the log10 scale.

(d) Summarize the effect of L based on the plots in (a)–(c).

Problem 7* Generate a length-10000 AR(2) signal. What are the selected orders by AIC, MDL, 10-fold cross-validation, and leave-one-out cross-validation? Note that you should use Yule–Walker equations for estimating the AR coefficients. For the purpose of cross-validation, you may consider cutting the AR signal into 50 non-overlapped segments of length 200, and consider each segment as a data point. Explain how you calculate $\hat{r}(k)$ during the cross-validation. Comment on the results you obtained. Repeat the problem for a length-10000 AR(10) signal. You may reuse the AR process related code from your project.