Statistical Signal Processing
5. The Levinson–Durbin Recursion

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Readings: Hayes §5.2; Haykin 4th Ed. §3.3

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Complexity in Solving Linear Prediction

Recall Augmented Normal Equations for linear prediction:

$$\underline{\text{FLP}} \quad \mathbf{R}_{M+1}\underline{a}_M = \begin{bmatrix} P_M \\ \underline{0} \end{bmatrix} \qquad\qquad \underline{\text{BLP}} \quad \mathbf{R}_{M+1}\underline{a}_M^{B*} = \begin{bmatrix} \underline{0} \\ P_M \end{bmatrix}$$

As $\mathbf{R}_{M+1}$ is usually non-singular, $\underline{a}_M$ may be obtained by inverting $\mathbf{R}_{M+1}$, or Gaussian elimination for solving equation array:

$\Rightarrow$ Computational complexity $O(M^3)$.

Note that these two equations are equivalent. Why?

Exploiting Structures in Matrix and LP Problem

Complexity in solving a general linear equation array:

- Method 1: invert the matrix, e.g., compute determinant of $\mathbf{R}_{M+1}$ matrix and the adjacency matrices

  $\Rightarrow$ matrix inversion has $O(M^3)$ complexity

- Method 2: use Gaussian elimination

  $\Rightarrow$ approximately $M^3/3$ multiplication and division

By exploring the Toeplitz structure of the matrix, Levinson–Durbin recursion can reduce complexity to $O(M^2)$

- M steps of order recursion, each step has a linear complexity w.r.t. intermediate order
- Memory use: Gaussian elimination $O(M^2)$ for the matrix, vs. Levinson-Durbin $O(M)$ for the autocorrelation vector and model parameter vector.

Levinson–Durbin Recursion

The **Levinson–Durbin recursion** is an order-recursion to efficiently solve linear systems with Toeplitz matrices, e.g., Augmented N.E. For M steps of order recursion, each step has a linear complexity w.r.t. intermediate order.

Goal: To solve $\underline{a}_m$ from the Augmented N.E., $\mathbf{R}_{m+1}\underline{a}_m = \begin{bmatrix} P_m \\ \underline{0} \end{bmatrix}$, where $\mathbf{R}_{m+1}$ is Toeplitz.

Plan: For N.E. at order $m+1$, we target to create an order recursion from order m.

Creating Order Update

First, create auxiliary vectors $\begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ \underline{a}_{m-1}^{B*} \end{bmatrix}$ using order-m vectors. Second, multiply from left using order-$(m+1)$ correlation matrix.

$$\text{\underline{FLP}} \quad \mathbf{R}_{m+1} \begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_m & \underline{r}_m^{B^*} \\ \underline{r}_m^{BT} & r(0) \end{bmatrix} \begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{R}_m \underline{a}_{m-1} \\ \underline{r}_m^{BT} \underline{a}_{m-1} \end{bmatrix} = \begin{bmatrix} P_{m-1} \\ \underline{0}_{m-1} \\ \Delta_{m-1} \end{bmatrix}, \text{ where } \Delta_{m-1} \triangleq \underline{r}_m^{BT} \underline{a}_{m-1}. \quad (1a)$$

$$\text{\underline{BLP}} \quad \mathbf{R}_{m+1} \begin{bmatrix} 0 \\ \underline{a}_{m-1}^{B*} \end{bmatrix} = \begin{bmatrix} r(0) & \underline{r}^H \\ \underline{r} & \mathbf{R}_m \end{bmatrix} \begin{bmatrix} 0 \\ \underline{a}_{m-1}^{B*} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{r}^H \underline{a}_{m-1}^{B*} \\ \mathbf{R}_m \underline{a}_{m-1}^{B*} \end{bmatrix} = \begin{bmatrix} \Delta_{m-1}^* \\ \underline{0}_{m-1} \\ P_{m-1} \end{bmatrix}. \quad (1b)$$

Third, poll these two equations together by (1a) + $\Gamma_m \times$ (1b):

Creating Order Update

$$\mathbf{R}_{m+1}\left(\begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix} + \Gamma_m \begin{bmatrix} 0 \\ \underline{a}_{m-1}^{B*} \end{bmatrix}\right) = \begin{bmatrix} P_{m-1} \\ \underline{0}_{m-1} \\ \Delta_{m-1} \end{bmatrix} + \Gamma_m \begin{bmatrix} \Delta_{m-1}^* \\ \underline{0}_{m-1} \\ P_{m-1} \end{bmatrix}.$$

Fourth, compare it to the order-$(m+1)$ N.E., $\mathbf{R}_{m+1}\underline{a}_m = \begin{bmatrix} P_m \\ \underline{0} \end{bmatrix}$. To obtain order update relationship, we need:

$$\begin{cases} \underline{a}_m \overset{\text{set}}{=} \begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix} + \Gamma_m \begin{bmatrix} 0 \\ \underline{a}_{m-1}^{B*} \end{bmatrix}, \\ \begin{bmatrix} P_m \\ \underline{0}_m \end{bmatrix} \overset{\text{set}}{=} \begin{bmatrix} P_{m-1} \\ \underline{0}_{m-1} \\ \Delta_{m-1} \end{bmatrix} + \Gamma_m \begin{bmatrix} \Delta_{m-1}^* \\ \underline{0}_{m-1} \\ P_{m-1} \end{bmatrix}. \end{cases}$$

Solving for Γ_m That Allows Order Update

$$\Rightarrow \begin{cases} P_m = P_{m-1} + \Gamma_m \Delta_{m-1}^* \\ 0 = \Delta_{m-1} + \Gamma_m P_{m-1} \end{cases}$$

$$\Gamma_m = -\frac{\Delta_{m-1}}{P_{m-1}} (= a_{m,m})$$

$$P_m = P_{m-1}\left(1 - |\Gamma_m|^2\right)$$

To ensure the prediction MSE $P_m \geq 0$, we require $|\Gamma_m|^2 \leq 1$.

P_m is non-increasing as we increase the order of the predictor, i.e., $P_m \leq P_{m-1}$, $\forall m > 0$.

Order Update Summary: Two Viewpoints of LD Recursion

Denote $\underline{a}_m \in \mathbb{C}^{(m+1)\times 1}$ as the tap weight vector of a forward-prediction-error filter of order $m = 0, ..., M$.

$a_{m-1,0} = 1$, $a_{m-1,m} \triangleq 0$, $a_{m,m} = \Gamma_m$ (*reflection coefficient*)

Order Update—Forward Prediction Viewpoint

$$a_{m,k} = a_{m-1,k} + \Gamma_m \, a^*_{m-1,m-k}, \; k = 0, 1, \ldots, m$$

Vector form: $$\underline{a}_m = \begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix} + \Gamma_m \begin{bmatrix} 0 \\ \underline{a}^{B*}_{m-1} \end{bmatrix} \quad (**)$$

Order Update—Backward Prediction Viewpoint

$$a^*_{m,m-k} = a^*_{m-1,m-k} + \Gamma^*_m \, a_{m-1,k}, \; k = 0, 1, \ldots, m$$

Vector form: $$\underline{a}^{B*}_m = \begin{bmatrix} 0 \\ \underline{a}^{B*}_{m-1} \end{bmatrix} + \Gamma^*_m \begin{bmatrix} \underline{a}_{m-1} \\ 0 \end{bmatrix}$$

(4) Reflection Coefficient Γ_m

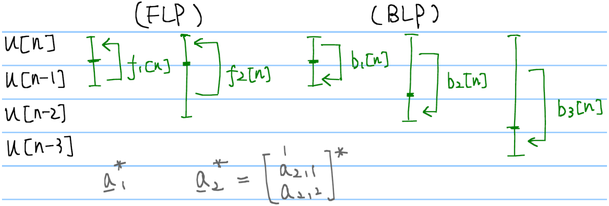
Let $P_0 = r(0)$ as the initial estimation error has power equal to the signal power, i.e., no regression is applied, we have

$$P_M = P_0 \cdot \prod_{m=1}^{M} (1 - |\Gamma_m|^2).$$

Question: Under what situation is $\Gamma_m = 0$?
i.e., increasing order won't reduce error.

Consider a process with Markovian-like property in 2nd order statistic sense (e.g. AR process) s.t. info of further past is contained in k recent samples.

Recall: Forward and Backward Prediction Errors



- $f_m[n] = u[n] - \hat{u}[n] = \underline{a}_m^H \underbrace{\underline{u}[n]}_{(m+1)\times 1}$
- $b_m[n] = u[n-m] - \hat{u}[n-m] = \underline{a}_m^{B,T} \underline{u}[n]$

(5) About Δ_m

One can show that the cross-correlation of <u>BLP error</u> and <u>FLP error</u> $\mathbb{E}\left[b_{m-1}[n-1]f^*_{m-1}[n]\right]$ is equal to Δ_{m-1}.

(Derive from the definition $\Delta_{m-1} \triangleq \underline{r}_m^{BT}\underline{a}_{m-1}$, and use definitions of $b_{m-1}[n-1], f^*_{m-1}[n]$ and orthogonality principle.)

Thus the reflection coefficient can be written as

$$\Gamma_m = -\frac{\Delta_{m-1}}{P_{m-1}} = -\frac{\mathbb{E}\left[b_{m-1}[n-1]f^*_{m-1}[n]\right]}{\mathbb{E}\left[|f_{m-1}[n]|^2\right]}$$

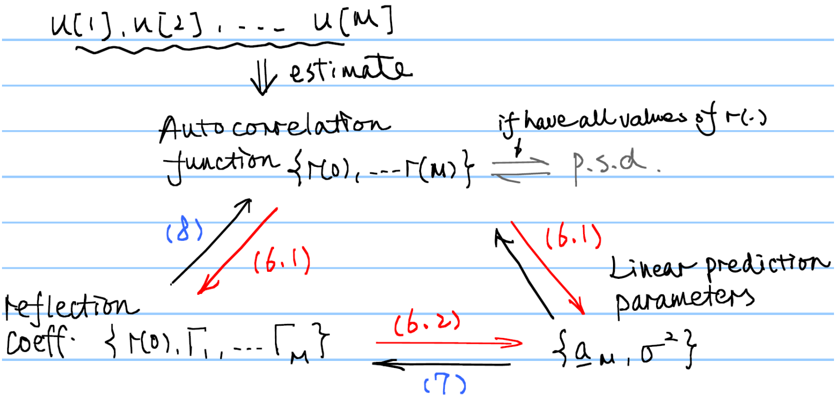
which is also the negative *partial correlation coefficient*.

Note: for the 0th order predictor, use the mean value, i.e., zero, as the estimate, s.t. $f_0[n] = u[n] = b_0[n]$,

$\therefore \Delta_0 = \mathbb{E}\left[b_0[n-1]f^*_0[n]\right] = \mathbb{E}\left[u[n-1]u^*[n]\right] = r(-1) = r^*(1)$

Preview: Relations of w.s.s and LP Parameters

For any w.s.s. process $\{u[n]\}$:



(6) Computing $\underline{a}_M$ and P_M by Forward Recursion

Case 1 : If we know the autocorrelation function $r(\cdot)$:

① $\Delta_0 = r(-1)$, $P_0 = r(0)$

② for $m = 1, \ldots M$ (order recursion)

$\quad \Gamma_m = -\frac{\Delta_{m-1}}{P_{m-1}}$

$\quad$ for $k = 1, \ldots m$ (diff predictor parameters for order-m)

$\quad\quad a_{m,k} = a_{m-1,k} + \Gamma_m a^*_{m-1,m-k}$

$\quad\quad$ (where $a_{m-1,0} = 1$; $a_{m-1,m} = 0$)

$\quad \Delta_m = r^{BT}_{m+1} \underline{a}_m$

$\quad P_m = P_{m-1}(1 - |\Gamma_m|^2)$

- # of iterations $= \sum_{m=1}^{M} m = \frac{M(M+1)}{2}$, comp. complexity is $O(M^2)$
- $r(k)$ may be estimated from time average of one realization of $\{u[n]\}$:

  $\hat{r}(k) = \frac{1}{N-k} \sum_{n=k+1}^{N} u[n]u^*[n-k], \; k = 0, 1, \ldots, M$

  (recall correlation ergodicity)

(6) Computing $\underline{a}_M$ and P_M by Forward Recursion

Case 2 : If we know Γ_1, Γ_2, ..., Γ_M and $P_0 = r(0)$, we can carry out the recursion for $m = 1, 2, \ldots, M$:

$$\begin{cases} a_{m,k} = a_{m-1,k} + \Gamma_m a^*_{m-1,m-k}, \ k = 1, \ldots, m \\ P_m = P_{m-1}\left(1 - |\Gamma_m|^2\right) \end{cases}$$

Note: $a_{m,m} = a_{m-1,m} + \Gamma_m a^*_{m-1,0} = 0 + \Gamma_m \cdot 1 = \Gamma_m$

(7) Inverse Form of Levinson-Durbin Recursion

Given the tap-weights $\underline{a}_M$, find the reflection coefficients $\Gamma_1, \Gamma_2, \ldots, \Gamma_M$:

Recall: $\begin{cases} \text{(FP) } a_{m,k} = a_{m-1,k} + \Gamma_m\, a^*_{m-1,m-k}, \ \ k = 0, \ldots, m \\ \text{(BP) } a^*_{m,m-k} = a^*_{m-1,m-k} + \Gamma^*_m\, a_{m-1,k}, \ \ a_{m,m} = \Gamma_m \end{cases}$

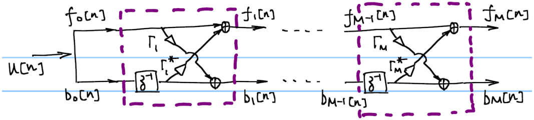
Multiply (BP) by Γ_m and subtract from (FP):

$$a_{m-1,k} = \frac{a_{m,k} - \Gamma_m a^*_{m,m-k}}{1 - |\Gamma_m|^2} = \frac{a_{m,k} - a_{m,m} a^*_{m,m-k}}{1 - |a_{m,m}|^2}, k = 0, \ldots, m-1$$

$\Rightarrow \Gamma_m = a_{m,m}, \ \Gamma_{m-1} = a_{m-1,m-1}, \ldots,$ i.e., From $\underline{a}_M \Rightarrow \underline{a}_m \Rightarrow \Gamma_m$

iterate with $m = M-1, M-2, \ldots$ *to lower order*

Lattice structure:



(8) Autocorrelation Function & Reflection Coefficients

Recall: The 2nd-order statistics of a stationary time series can be represented in terms of autocorrelation function $r(k)$, or equivalently the power spectral density by taking DTFT.

Another way is to use $\{r(0), \Gamma_1, \Gamma_2, \ldots, \Gamma_M\}$.

To find the relation between them, recall:

$\Delta_{m-1} \triangleq \underline{r}_m^{BT} \underline{a}_{m-1} = \sum_{k=0}^{M-1} a_{m-1,k} r(-m+k)$ and $\Gamma_m = -\frac{\Delta_{m-1}}{P_{m-1}}$

$\Rightarrow -\Gamma_m P_{m-1} = \sum_{k=0}^{m-1} a_{m-1,k} r(k-m)$, where $a_{m-1,0} = 1$.

(8) Autocorrelation Function & Reflection Coefficients

1. $r(m) = r^*(-m) = -\Gamma_m^* P_{m-1} - \sum_{k=1}^{m-1} a_{m-1,k}^* r(m-k)$

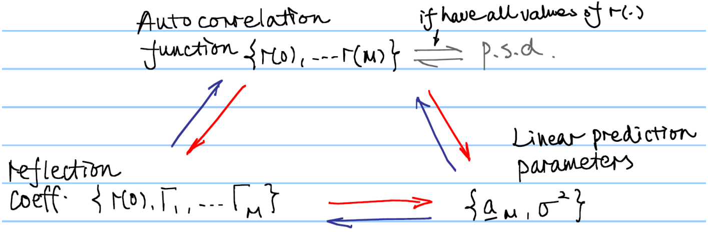
   $r(1), \ldots, r(M)$ can be generated iteratively. Note that $\underline{a}_m$ can be found using $r(0), \Gamma_1, \Gamma_2, \ldots, \Gamma_M$ by (6.2).

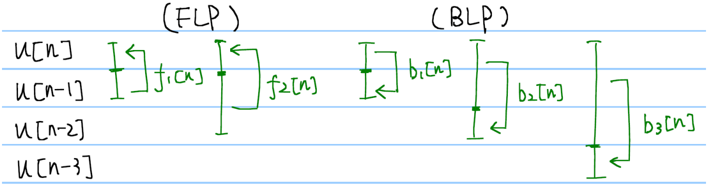
2. Recall if $r(0), \ldots, r(M)$ are given, we can get $\underline{a}_m$ by (6.1). So $\Gamma_1, \ldots, \Gamma_M$ can be obtained iteratively: $\Gamma_m = a_{m,m}$.

3. These facts imply that the reflection coefficients $\{\Gamma_k\}$ can uniquely represent the 2nd-order statistics of a w.s.s. process.

Summary

Statistical representation of w.s.s. process





Detailed Derivations/Examples

Example of Forward Recursion Case 2

e.g. (case 2). Given $\Gamma_1, \Gamma_2, \Gamma_3$ and $P(0)$, find $\underline{a}_3$ and P_3 of a prediction-error filter of order 3.

(0) $P_0 = r(0)$

(1) $m=1$: $a_{1,0} = 1$; $a_{1,1} = \Gamma_1$; $a_{1,2} = 0$; $P_1 = P_0(1 - |\Gamma_1|^2)$

(2) $m=2$: $a_{2,0} = 1$; $a_{2,1} = a_{1,1} + \Gamma_2 a_{1,1}^* = \Gamma_1 + \Gamma_2 \cdot \Gamma_1^*$ (used in §2.5.4. for inverse filtering)

$a_{2,2} = \Gamma_2$

$P_2 = P_1(1 - |\Gamma_2|^2)$

(3) $m=3$: $a_{3,0} = 1$; $a_{3,1} = a_{2,1} + \Gamma_3 a_{2,2}^* = \Gamma_1 + \Gamma_2 \Gamma_1^* + \Gamma_3 \cdot \Gamma_2^*$

$a_{3,2} = a_{2,2} + \Gamma_3 a_{2,1}^* = \Gamma_2 + \Gamma_3 \Gamma_1^* + \Gamma_1 \Gamma_2^* \Gamma_3$

$a_{3,3} = \Gamma_3$

$P_3 = P_2(1 - |\Gamma_3|^2)$

Proof for Δ_{m-1} Property

Proof:

$$\Delta_{m-1} = \Gamma_m^{BT} a_{m-1} = [r(-m), \dots r(-1)]\, a_{m-1}$$
$$= E[u^*[n]\, u_m^{BT}[n-1]]\, a_{m-1}$$
$$= E[u^*[n]\, u_m^{BT}[n-1]\, a_{m-1}]$$
$$= E[u^*[n]\, (u_m^{T}[n-1]\, a_{m-1}^{B})]$$
$$= E[u^*[n]\, b_{m-1}[n-1]]$$
$$= E[f^*_{m-1}[n]\, b_{m-1}[n-1]]$$

Recall

① $\Gamma_M = \begin{bmatrix} r(-1) \\ \vdots \\ r(-M) \end{bmatrix}$

② $r(-k) = E[u[n-k]\, u^*[n]] = (E[u[n]\, u^*[n-k]])^*$

③ $u_m[n-1] = \begin{bmatrix} u[n-1] \\ \vdots \\ u[n-m] \end{bmatrix}$

④ $b_{m-1}[n-1] = \sum_{k=0}^{m-1} a_{m-1,m-1-k}\, u[n-1-k] = a_{m-1}^{B}\, u_m[n-1]$

⑤ $a_{m-1,0} = 1$

$$f_{m-1}[n] = \sum_{k=0}^{m-1} a^*_{m-1,k}\, u[n-k] = u[n] + \sum_{k=1}^{m-1} a^*_{m-1,k}\, u[n-k]$$

⑥ $b_{m-1}[n-1] \perp \{u[n-1], \dots u[n-m+1]\}$

Haykin's 4th Ed. (P152)

* partial correlation (PARCOR) coeff. between $f_{m-1}[n]$ and $b_{m-1}[n-1]$:

$$\rho_m \triangleq \frac{E[b_{m-1}[n-1]\, f^*_{m-1}[n]]}{\left(E[|b_{m-1}[n-1]|^2]\, E[|f_{m-1}[n]|^2]\right)^{1/2}} \overset{\text{for w.s.s.}}{=} \frac{\Delta_{m-1}}{P_{m-1}} = -\Gamma_m$$

Recall $E[|f_m[n]|^2] = E[|b_m[n]|^2] = P_m$